Introduction

Fair division is the process of dividing a set of goods among several people in a way that is "fair". However, what exactly we mean by fairness is deceptively complex. We'll explore different notions of fairness in depth throughout this Power Round.

We'll begin with the canonical motivating example, called the *Cake-Cutting Problem.* Suppose you and your friend wish to split a cake. If the cake is homogeneous (all the same), then it is clearly most fair to split the cake in half, so you each receive an equal share. However, suppose half of the cake contains cherries, while the other half does not, and additionally suppose that you really like cherries, while your friend does not. In this case, splitting the cake equally is no longer obviously the best solution. Since you prefer cherry cake to regular cake while your friend does not, splitting the cake so you get the cherry half and your friend gets the rest is intuitively better than splitting both halves equally. These notions will be formalized shortly.

Definitions

All fair division problems share a few common features. First, they contain a set of n **players** which we will number Player 1 to Player n, and a finite set of **goods**, G, that we wish to divide among the players.

The elements of G may be one of three different types. Some goods, such as a car or a dog, are **indivisible**, which means they can only be assigned to a single player. Other goods, like money or cake are **divisible**, which means they can be divided among multiple players. We further distinguish between **homogeneous** divisible goods like money whose parts are indistinguishable from each other, and **heterogeneous** goods like cake, whose parts may not all be equivalent. In the example above, the half-cherry cake is heterogeneous and divisible, because it can be split among players, but its parts are distinguishable, since some will have more cherries than others.

Define a **division** to be an allocation of goods to players. We can denote divisions by listing the goods assigned to each player. For example, one possible division of the set of goods $G = \{A, B, C\}$ among two players is $(1 : (A, B),\ 2 : (C))$, which denotes that Player 1 receives A and B, while Player 2 receives C. If the goods are divisible and homogeneous, we can add a fraction to the good to denote the portion received by each player. For example, $(1 : (A, B, \frac{1}{3}C),\ 2 : (\frac{2}{3}C))$ denotes that Player 1 receives A, B, and one third of C, while Player 2 receives the remaining two thirds of C. Note that this notation isn't well-defined for heterogeneous goods, since if A is heterogeneous then "half of A" could mean any half, and they are not all necessarily equivalent.

Next, to model players' preferences, we assume each player has a **value function** that describes how much they value the goods in G. Define v_i to be player i's value function, and let $v_i(A)$ be the value player i places on the goods in set A ($A \subseteq G$). Value functions have the following properties:

i. Values placed on goods by players are always nonnegative real numbers.

ii. For each player i, $v_i(\{\}) = 0$ and $v_i(G) = 1$ (that is, all players place the same total value on all of G).

iii. Players' values are **additive**. That is, $v_i(A \cup B) = v_i(A) + v_i(B) - v_i(A \cap B)$ for any sets of goods A and B.

iv. Divisible, *homogeneous* goods are valued linearly. For example, if $v_i(\{g\}) = x$, and g is a homogeneous good, then player i would assign value $\frac{x}{2}$ to any half of g, $\frac{x}{5}$ to any fifth of g, etc. Note that this is *not* necessarily the case for heterogeneous goods, for example we've seen that not all cuts of a cake into halves will be valued equally by all players.

Hence, all players assign the same total value to all of G, and a player will never prefer *not* receiving a good g to receiving it. Additionally, the value function of any player can be enumerated by listing the value they place on each element of G. Note, however, that value functions can't necessarily be enumerated for all possible divisions of a heterogeneous good, since there may be uncountably many distinguishable divisions.

Define the **total value** that a player receives in a particular division to be the sum of values that player places on all goods they receive. For example, if $v_1(\{g_a\}) = 0.1$, $v_1(\{g_b\}) = 0.2$, and some division assigns goods g_a and g_b to Player 1, then the total value Player 1 receives from this division will be $v_1(\{g_a, g_b\}) = v_1(\{g_a\}) + v_1(\{g_b\}) = 0.1 + 0.2 = 0.3$. Note that total values depend on each player's *own* value functions, so the total value Player 2 receives from g_a and g_b is not necessarily the same as the total value Player 1 receives.

Finally, we assume that players are always **rational** and **honest**. This just means that players will always prefer to maximize the total value they receive, they will never prefer to receive less just to spite another player, and they will never lie about their true preferences until Problem 9.

Proportionality

A division is **proportional** if each of the n players receives a total value of at least $\frac{1}{n}$ according to *their own* value function.
A division is **super proportional** if each player receives total value *greater than* $\frac{1}{n}$ according to their own value function.

1. (a) [**6**] Suppose we wish to divide $G = \{A, B, C\}$, among two players (Player 1 and Player 2), and A, B, and C are indivisible. For each of the following, use the given value functions to find a division that satisfies proportionality. If a proportional division doesn't exist, write "no proportional division".

i.

	v_1	v_2
A	0.4	1.0
B	0.3	0
C	0.3	0

ii.

	v_1	v_2
A	0.6	0.6
B	0.4	0
C	0	0.4

iii.

	v_1	v_2
A	0.3	0.3
B	0.5	0.5
C	0.2	0.2

(b) [**4**] Suppose that all elements of G are divisible and homogeneous. Prove that a proportional division always exists.

(c) [**15**] Suppose that all elements of G are divisible and homogeneous. Prove that a super proportional division exists if and only if not all players have the same value function.

Solution to Problem 1:

(a) i. There is a unique proportional division here. Player 1 gets $\{B, C\}$ so his total value is $v_1(B) + v_1(C) = 0.3 + 0.3 = 0.6 \geq 0.5$ and Player 2 gets $\{A\}$ so his total value is $v_2(A) = 1.0 \geq 0.5$.

ii. No proportional division exists.

iii. There are two possible proportional divisions. Either Player 1 gets $\{A, C\}$ and Player 2 gets $\{B\}$ or Player 1 gets $\{B\}$ and Player 2 gets $\{A, C\}$.

(b) We give a constructive argument. Let G consists of k goods $\{g_1, g_2, \ldots, g_k\}$ and suppose there are n players. Then divide the goods so that each player gets $\frac{1}{n}$th of each good. The total value of player i is then $\sum_{j=1}^{k} v_i(\frac{1}{n}g_j) = \sum_{j=1}^{k} \frac{1}{n}v_i(g_j) = \frac{1}{n}\sum_{j=1}^{k} v_i(g_j) = \frac{1}{n}$, so each player gets a total value of at least $\frac{1}{n}$.

(c) Suppose that there are n players, all of whom have the same value function v, and let $G = \{g_1, g_2, ..., g_m\}$ be the set of goods they wish to divide. Let $G_i = \{g_{1i}, g_{2i}, \ldots, g_{mi}\}$ be the division of goods the ith player gets. We wish to show that a super proportional division does not exist. For contradiction, assume that a super proportional division does exist, i.e. $v(G_i) > 1/n$ for all i. If we sum up over all $v(G_i)$ we get $v(G_1) + v(G_2) + \cdots + v(G_n) = v(g_{11}) + v(g_{21}) + \cdots + v(g_{m1}) + \cdots + v(g_{mn}) = v(g_1) + \cdots + v(g_n) = 1 > 1$, contradiction.

For the other direction, we induct on n. For the base case, let player 1 have value function v_1, player 2 have value function v_2, and let $G = \{g_1, g_2, \ldots, g_m\}$ be the set of goods they wish to divide. Give both players the division $G_1 = G_2 = \{0.5g_1, \ldots, 0.5g_m\}$. Then $v_1(G_1) = v_2(G_2) = 1/2$. Because $v_1 \neq v_2$, there exists some goods g_i, g_k such that $v_1(g_i) > v_2(g_i)$ and $v_1(g_j) < v_2(g_j)$. Player 1 then "trades" player 2 by giving away $0.5g_{j1}$ in exchange for $0.5g_{i2}$. Call these new divisions G_1' and G_2'. Then $v_1(G_1') = 1/2 + v_1(g_i/4) - v_1(g_j/4) > 1/2$ and $v_2(G_2') = 1/2 + v_2(g_j/4) - v_1(g_i/4) > 1/2$, so we have found a super proportional division of G over 2 players. This completes the base case.

For the inductive step, assume that we have a super proportional division of G over n players. We wish to show that we can form a super proportional division over $n+1$ players. We first redistribute the goods by giving player $n+1$ the goods $1/(n+1)(G_1 + \cdots + G_n)$. Because $G_1 + \cdots + G_n = G$, we have $v_{n+1}(1/(n+1)G) = 1/(n+1)$. Furthermore, by our inductive hypothesis, $v_k(n/(n+1)G_k) = n/(n+1)v_k(G_k) > n/(n+1) \cdot 1/n = 1/(n+1)$. If we can find a "trade" such that $v_{n+1}(G_{n+1}) > 1/(n+1)$ we are done. By our assumption, there exists some v_k such that $v_{n+1} \neq v_k$, which means there exist some goods g_i, g_j such that $v_k(g_i) > v_{n+1}(g_i)$ and $v_k(g_j) < v_{n+1}(g_j)$. We then "trade" in a similar fashion as in our base case. Let $t = 0.5\min(g_{ik}, g_{jk}, g_{i(n+1)}, g_{j(n+1)})$. Give player $ktg_{i(n+1)}$ and give player $(n+1)tg_{jk}$. Call these new divisions G_k' and G_{n+1}'. Then $v_k(G_k') > 1/(n+1) + v_k(tg_i) - v_k(tg_j) > 1/(n+1)$ and $v_{n+1}(G_{n+1}') = 1/(n+1) + v_{n+1}(tg_j) - vn + 1(tg_i) > 1/(n+1)$, meaning we have found a super proportional division of G over $n+1$ players. This completes the inductive step, so a super proportional division exists if not all players have the same value function.

Envy-Free Divisions

A division is **envy-free** if every player believes, from their own perspective, that the total value they received is at least as high as any other players'. That is, if each player i is assigned goods G_i, then for each i and all $j \neq i$, $v_i(G_i) \geq v_i(G_j)$. In other words, no player is envious of any other.

For example, with two players and the value functions given in 1.a.i, the division $(1 : (B), 2 : (A, C))$ is not envy-free, because Player 1 received a total value of 0.3, while Player 1 perceives Player 2 to have received a total value of 0.7. Hence, Player 1 will be envious of Player 2.

2. (a) **[6]** Suppose we wish to divide the set $G = \{A, B, C, D\}$ among three players, and A, B, C, and D are indivisible. For each of the following, use the given value functions to find a division that is envy-free. If an envy-free division doesn't exist, write "no envy-free division".

i.

	v_1	v_2	v_3
A	0.25	0.3	0.5
B	0.25	0.4	0
C	0.25	0.2	0
D	0.25	0.1	0.5

ii.

	v_1	v_2	v_3
A	0.25	0.3	0.1
B	0.25	0.3	0.2
C	0.25	0.2	0.3
D	0.25	0.2	0.4

(b) **[4]** Prove that every envy-free division is also proportional.

(c) **[5]** With two players, is every proportional division also envy-free? Prove, or disprove by finding a counterexample.

(d) **[6]** With three players, is every proportional division also envy-free? Prove, or disprove by finding a counterexample.

Solution to Problem 2:

(a) i. There are two possible envy-free divisions.

- Player 1 is allocated $\{C, D\}$, Player 2 is allocated $\{B\}$, and Player 3 is allocated $\{A\}$.
- Player 1 is allocated $\{A, C\}$, Player 2 is allocated $\{B\}$, and Player 3 is allocated $\{D\}$.

ii. There is no envy-free division.

(b) It is easier to prove the contrapositive, that if a division is not proportional, then it cannot be envy-free. Suppose a division among n players is not proportional. Then some player i receives total value less than $\frac{1}{n}$. It follows that using that player's value function v_i, the total value of the remaining goods is strictly greater than $\frac{n-1}{n}$ and is divided amongst the other $n-1$ players. Therefore, at least one of the other players must have total value greater than $\frac{1}{n}$ as measured by v_i so player i will be envious of the player with total value greater than $\frac{1}{n}$. Thus, if a division is not proportional, it cannot be envy-free.

(c) It is true that with two players, every proportional division is also envy-free. Suppose we have a proportional division of n goods between two players. Player 1 receives a total value of at least $\frac{1}{2}$ according to his value function, so the total value of the goods Player 2 received is at most $1 - \frac{1}{2} = \frac{1}{2}$ so Player 1 is not envious of Player 2. Similarly, Player 2 cannot be envious of Player 1.

(d) It is *not* true that with three players, every proportional division is envy-free. Consider the following counter example:

	v_1	v_2	v_3
A	1/3	1/6	1/2
B	1/2	1/3	1/6
C	1/6	1/2	1/3

and consider the division where Player 1 gets $\{A\}$, Player 2 gets $\{B\}$ and Player 3 gets $\{C\}$. Then each player has total value $\frac{1}{3}$ so the division is proportional. However, Player 1 is envious of Player 2 since from Player 1's perspective, Player 2 has a total value of $\frac{1}{2}$.

Efficiency

A division d_1 **dominates** another division d_2 if at least one player receives strictly greater total value in d_1 than d_2, and no player receives less total value in d_1 than d_2.

A division is **Pareto efficient** if it is not dominated by any other division. Equivalently, a division is Pareto efficient if all other divisions either assign identical total values to each player, or assign a lower total value to at least one player.

3. (a) **[6]** Suppose $G = \{A, B, C, D\}$ is a set of goods to be divided among two players, where A, B, C, and D are each indivisible. Given the following value functions for the players, list all Pareto efficient divisions, and indicate whether or not each one is proportional.

	v_1	v_2
A	0.3	0.1
B	0.3	0.2
C	0.2	0.3
D	0.2	0.4

(b) **[6]** Now, suppose $G = \{A, B, C\}$, A, B, and C are indivisible, and we wish to divide G among two players. Choose value functions for the two players, and find a proportional division that is not Pareto efficient.

(c) **[10]** Let G be a set of divisible, homogeneous goods. Prove that a division between two players is *not* Pareto efficient if and only if there are goods g and g' in G such that a trade between two players of a part of g for a part of g' yields a division that dominates the given one.

Solution to Problem 3:

(a)

Player 1	Player 2	Is Proportional?
{A,B,C,D}	{}	No
{A,B,C}	{D}	No
{A,B}	{C,D}	Yes
{A}	{B,C,D}	No
{}	{A, B,C,D}	No

(b) There may be other possible answers, but here is one possibility:

	v_1	v_2
A	0.5	0.5
B	0	0.5
C	0.5	0

with the division where Player 1 is allocated $\{A\}$ and Player 2 is allocated $\{B, C\}$. The total value of both players is 0.5 so this is proportional division, but is not Pareto efficient because the division where Player 1 is allocated $\{A, C\}$ and Player 2 is allocated $\{B\}$ dominates this.

(c) Suppose a division D that assigns goods A to Player 1 and goods B to Player 2 is not Pareto efficient. Then there is another Division D' that is Pareto efficient and dominates D. Consider the differences between D and D'. It must consist of some sets of goods (or fractions of goods) $A_0 \subset A$ and $B_0 \subset B$ such that Player 1 trading A_0 for Player 2's B_0 results in D'. In particular, since D' dominates D, this means $v_1(B_0) \geq v_1(A_0)$ and $v_2(A_0) \geq v_2(B_0)$ and at least one of these inequalities is strict.

Split A_0 into disjoint sets $A_1, A_2, \ldots A_m$ such that $A_0 = A_1 \cup A_2 \cup \cdots \cup A_m$ and each A_i for $i \geq 1$ is a subset of a single good. Since $v_2(A_0) \geq v_2(B_0)$, we know that Player 2 can split B_0 into disjoint sets $B_1, B_2, \ldots B_m$ such that $B_0 = B_1 \cup B_2 \cup \cdots \cup B_m$ and a trade of A_i for B_i leaves Player 2 no worse off than before. Note that each B_i for $i \geq 1$ is not necessarily a subset of a single good.

Because we know D is not efficient and preferences are additive, we know that there exists at least one $i \geq 1$ such that Player 1 finds a trade of A_i for B_i strictly preferable to D. Choose such an i and write B_i as the disjoint union of sets $B_{i1}, \ldots, B_{in}$ such that each B_{ij} is a subset of a single good, and as before split A_i into corresponding sets such that Player 1 is no worse off than before after trading B_{ij} for A_{ij} for any j. By the same logic as above, there must be some j such that Player 2 strictly prefers trading B_{ij} for A_{ij} to D.

Since A_i was already a subset of a single good, both B_{ij} and A_{ij} are subsets of a single good, Player 1 is no worse off after trading A_{ij} for B_{ij}, and Player 2 is better off after trading B_{ij} for A_{ij}. Hence, performing this trade yields a division that dominates D, as desired.

The Price of Proportionality

We now consider another metric often used to judge divisions.

Define the **welfare** of a division to be the sum of the total values each player receives. For example, if there are two homogeneous divisible goods A and B, and Player 1 values A at 1 (and B at 0) while Player 2 values B at 1 (and A at 0), then giving A to Player 1 and B to Player 2 gives them each a total value of 1, for a welfare of $1 + 1 = 2$, while dividing both goods equally between the players will give them each a total value of 0.5, for a welfare of 1. Hence, while both of these divisions are proportional and envy-free, the former provides the higher welfare (note that the latter is also not Pareto efficient).

Maximizing welfare seems like a useful thing to aspire to, but it's often at odds with our other notions of fairness. As we'll see we often must sacrifice some welfare to achieve a proportional division. This is particularly true for divisions involving a large number of players. For a given scenario (set of goods, players, and player value functions), define the **utilitarian price of proportionality**, or **UPOP**, to be the ratio between the maximal welfare, and the maximal welfare in a proportional division.

4. (a) **[5]** Prove that any division that maximizes welfare is Pareto efficient.

   (b) **[5]** Given the value functions from 2.a.i for indivisible goods A, B, C, and D, compute

      i. the division that maximizes welfare.

      ii. the proportional division with the greatest welfare.

      iii. the UPOP of this scenario.

   (c) **[10]** Suppose that we wish to divide a set of divisible goods among 25 players. Using 25 goods, describe a scenario with a UPOP of at least 2.5.

(d) **[18]** Suppose again that we wish to divide a set of divisible goods among 25 players. Prove that the UPOP of any such scenario is ≤ 9.

Solution to Problem 4:

(a) Suppose for contradiction, that there exists a division D that maximizes welfare but is not Pareto efficient. Then there must be some other division D' that dominates the current division. By definition, this implies that the total value of each player under division D' is greater than or equal to the total value of that player under division D, and that for some i, the total value of Player i under division D' is strictly greater than the total value of Player i under division D. But this is a contradiction, since the welfare of D' must be greater than D, but it was assumed that D maximized welfare.

(b) i. The division that maximizes welfare allocates to Player 1 $\{C\}$, to Player 2 $\{B\}$, and to Player 3 $\{A, D\}$. The welfare is 1.65.

ii. There are two proportional division with the greatest welfare. One possible division allocates to Player 1 $\{A, C\}$, to Player 2 $\{B\}$, and to Player 3 $\{D\}$. The other proportional division is where Player 1 gets $\{C, D\}$, Player 2 gets $\{B\}$, and Player 3 gets $\{A\}$. In either case, the welfare is 1.4.

iii. The UPOP is $\frac{1.65}{1.4}$.

(c) One solution is as follows: Let the goods be denoted $G = \{g_1, g_2, \ldots, g_{25}\}$ and let players $6, 7, \ldots, 25$ value each good equally. That is, $v_i(g_j) = \frac{1}{25}$ for $6 \leq i \leq 25$ and $1 \leq j \leq 25$. For the remaining 5 players, we will have them value 5 goods equally and the remaining goods at 0. That is, $v_i(g_j) = \frac{1}{5}$ for $1 \leq i \leq 5$ and $5(i-1) \leq j \leq 5i$.

Then a division that maximizes welfare will give Players 1 through 5 the goods that they value for $\frac{1}{5}$ for a welfare of 5. Any proportional division must give Players 6 through 25 at least one good and hence Players 1 through 5 at most one good. Therefore, any proportional division has a welfare of at most $20 \cdot \frac{1}{25} + 5 \cdot \frac{1}{5} = 1.8$. Hence, the UPOP is $\frac{5}{1.8} \geq 2.5$.

(d) Begin with any division D, and split the players into those who received total value at least $\frac{1}{5}$ (the "fortunate") and those who received total value less than $\frac{1}{5}$.

If fewer than 5 players are fortunate, then the welfare is at most $5 \cdot 1 + 20 \cdot \frac{1}{5} = 9$. Since the welfare of any proportional division is at least 1, this means the UPOP in this case is at most 9, as desired.

Now, suppose at least 5 players are fortunate. We will provide a construction that transforms D into a proportional division with at least one fifth the welfare, from which it follows that the UPOP in this case is at most 5.

First, we define a prodedure called *diminishing*. Suppose we have n players. First, let a player i choose a set of goods A he believes has value $\frac{1}{n}$. Each of the diminishing players takes turns **diminishing** A by removing part of it until they believe the remaining part, combined with any goods they have already been allocated, has value $\frac{1}{n}$ (or doing nothing if they believe the total value is less than $\frac{1}{n}$). After each player has had a chance to diminish A, the last player to diminish receives what is left of A, and the remainder is returned to Player i. As a result, the last diminisher is guaranteed to finish with total value at least $\frac{1}{n}$, and it can easily be seen that this procedure can be repeated, since all other players were unsatisfied with the fraction received by the last diminisher.

Taking turns, each fortunate player cuts a part of their share with total value to them of $\frac{1}{25}$, and lets the unfortunate players diminish it. Repeat this until each unfortunate player has total value at least $\frac{1}{25}$, as they would expect in a proportional division. This will require at most 20 steps, one for each unfortunate player. At the end, each unfortunate player will have total value at least $\frac{1}{25}$, and each fortunate player will have to give away at most $4 \cdot \frac{1}{25} = \frac{4}{25}$ of their total value, so since they started with $\frac{1}{5}$ they will end with at least $\frac{1}{5} - \frac{4}{25} = \frac{1}{25}$. Hence, the resulting division is proportional.

Since each unfortunate player ends with at least as much total value as they had before, and each fortunate player ends with at least one fifth of their original total value, the welfare of the final proportional division is at least one fifth the original welfare, so the UPOP in this case is at most 5.

Hence, in both cases the UPOP is ≤ 9, as desired.

Divide and Choose

One common solution to the cake-cutting problem described in the introduction and other two-player fair division problems is a procedure called **Divide and Choose**. In Divide and Choose, one player cuts the cake into any two pieces that they believe are of equal value. Then the second player chooses which of the pieces they prefer, and finally the first player receives the remaining piece.

More generally, Divide and Choose can be used for many fair division problems with two players. One player chooses a division of G into two sets, each of which would give him equal total value. Then, the other player chooses which set of goods she prefers, and the first player receives the remaining set. Note that the procedure doesn't always work if G contains an indivisible element, since in that case the first player can't necessarily divide G into two halves of equal value.

5. (a) [**8**] Let G be a cake with three divisible, homogeneous regions, C, V, and S (corresponding to chocolate, vanilla, and strawberry). Suppose Player 1 and Player 2 have the following value functions:

	v_1	v_2
C	0.2	0.3
V	0.3	0.4
S	0.5	0.3

If both players follow the procedure exactly, which of the following divisions could be the result of Divide and Choose? Note that either player could be the divider. For each case, write whether it is possible or not.

i. $\{1 : (S), 2 : (C, V)\}$
ii. $\{1 : (C, V), 2 : (S)\}$
iii. $\{1 : (V, \frac{1}{2}S), 2 : (C, \frac{1}{2}S)\}$
iv. $\{1 : (\frac{1}{2}V, S), 2 : (C, \frac{1}{2}V)\}$

(b) [**4**] Prove, or disprove by finding a counterexample, that Divide and Choose always results in an envy-free division.

(c) [**6**] Prove, or disprove by finding a counterexample, that Divide and Choose always results in a Pareto efficient division.

(d) [**6**] Consider the following procedure, which attempts to generalize Divide and Choose to three players: Player 1 first creates a division of G of the goods into three sets, each of which would give him equal total value. Then, Player 2 chooses the set from the three that he prefers, Player 3 chooses the set from the remaining two that she most prefers, and finally Player 1 receives the last remaining set. Is this procedure envy-free? Prove, or disprove by finding a counterexample.

Solution to Problem 5:

(a) i. Possible

ii. Not possible (player 1 makes division, but player 2 would not choose S).

iii. Not possible

iv. Possible

(b) It is true that Divide and Choose always results in an envy-free division. Player 1 divides the goods into two sets that give him equal value so no matter which set he receives, he will receive at least 0.5 total value. So from Player 1's point of view, Player 2 will receive at most 0.5 total value and hence Player 1 will not be envious of Player 2. Player 2 will pick the set of goods she prefers so by definition of the procedure, Player 2 will not be envious of Player 1. Thus, Divide and Choose always results in an envy-free division.

(c) It is not true that Divide and Choose always results in a Pareto efficient division. There are many counterexamples. One possible counterexample is given with 4 goods A, B, C, D and the following value functions

	v_1	v_2
A	0.25	0.1
B	0.25	0.3
C	0.25	0.2
D	0.25	0.4

Player 1 may divide the goods into sets $\{A, B\}$ and $\{C, D\}$ in which case Player 2 would choose to have $\{C, D\}$. However, this is not Pareto efficient, because the division where Player 1 receives $\{A, C\}$ and Player 2 receives $\{B, D\}$ dominates the division that result from Divide and Choose.

(d) This procedure is not envy-free. For example, consider three objects A, B, and C. Player 1 values them equally, but Players 2 and 3 value A at 1 and B and C at 0. Player 1 values them equally, so he can divide up the objects into three sets, one in each set. Player 2 and Player 3 place all of their value on item A. So Player 2 will take the set containing A, leaving Player 3 with two worthless options. Player 3 will necessarily envy player 2.

Moving Knives

The special case where $G = \{g\}$ contains a single heterogeneous element (for example, if g is a cake) is common, and a number of fair division procedures have been developed specifically for it. One class of such procedures is called the **Moving Knife Procedures**.

For this problem, we make a few simplifying assumptions. First, let the real numbers in the interval $(0, 1)$ correspond to horizontal position across the width of g, which we imagine is similar to a cake. Restrict the cuts we can make in the cake to cuts at positions $x \in (0, 1)$. Then we can describe a division of $G = \{g\}$ by the numeric positions of the cuts we make on g, and the allocation of the resulting pieces to players. Additionally, this restriction allows us to model each player's value function as a nonnegative function on the interval $[0, 1]$ such that if f is a value function then $\int_0^1 f(x)\,dx = 1$ and $\int_a^b f(x)\,dx$ is the value the player will place on the piece between cuts at a and b. Finally, we assume that these value functions are continuous, so these integrals are well-behaved.

6. (a) **[4]** The two-player Moving Knife Procedure proceeds as follows: A "knife" is held over g at position 0, and slowly swept across so the numeric position of the knife gradually increases. At any point, if either player believes that the portion of g to the left of the knife has half the value of all of g, they call a stop to the procedure, and g is cut at that point. Then, the player who stopped the knife receives the left half and the other player receives the right half. Prove that, like Divide and Choose, this procedure is results in an envy-free division.

   (b) **[8]** A nice advantage of this procedure is that it easily generalizes to more than two people. Consider the following modification for n people: The knife is swept as before, but as soon as any of the n players believe the portion of g to the left of the knife has value $\frac{1}{n}$ of the whole, they call stop. Then, g is cut at that point, and the player who called stop receives the piece to the left of the knife. Then, the entire procedure is repeated with the remainder of g and the remaining $n - 1$ players. Prove that this procedure results in a proportional division.

   (c) **[8]** Consider the following variant of the two-player moving knife procedure: Player 1 holds two knives. The first is initially at the left edge of g, and the second is placed at the line that Player 1 believes splits g into two halves of equal value. Then, Player 1 sweeps both knives slowly to the right, such the portion of g between the knives remains exactly half the value of all of g. As soon as Player 2 agrees that the portion between the knives is half the value of all of g, she tells Player 1 to stop. Then, g is cut at the position of the knives, the center piece is given to Player 2, and the remainder is given to Player 1.

      i. Prove that this procedure always terminates.

      ii. Prove that this procedure results in a division where both players believe that they and the other player received a total value of exactly $\frac{1}{2}$. Such a division is called an **exact** division, and is clearly also envy-free and proportional.

Solution to Problem 6:

(a) Without loss of generality, suppose Player 1 was the person who called stop to the procedure. Then the portion of the goods Player 1 receives has total value at least 0.5 and hence the remaining portion of the goods has total value at most 0.5 so Player 1 is not envious of Player 2. Since Player 1 is the person to call stop to the procedure, the portion to the left of the knife must have total value at most 0.5 to Player 2. Player 2 receives everything to the right of the knife which she therefore must have total value at least 0.5 to her. Thus, Player 2 is also not envious of Player 1 so the Moving Knives procedure is envy-free.

(b) Each of the first $n-1$ players who call stop only call stop once they believe the portion to the left of the knife has value $\frac{1}{n}$ so each of them will receive total value at least $\frac{1}{n}$. Now consider the remaining player. During each step of the procedure, stop is called before she thinks the piece to the left of the knife has total value $\frac{1}{n}$. Thus, after $n-1$ calls, the total value of the pieces taken is worth at most $\frac{n-1}{n}$ to the remaining player, so the portion of the goods given to the remaining player also has total value at least $\frac{1}{n}$. Thus, this generalized Divide and Choose procedure must result in a proportional division.

(c) i. Assume that at time $t=0$, player 1 holds the knife at 0 and b_0, such that 1 values the cake between those two positions at 1/2. Say at time t, the knives are at positions $a_t < b_t$ such that 1 values the cake between them at 1/2, and assume $a_1 < b_1 = 1$. Now, at time 0 and at time 1, the cake would be split in the same way, but the pieces would go to the other player. Now, since player 2's value of the pieces add to 1, either both pieces are worth 1/2 to player 2, in which case 2 will stop the procedure immediately, or 2 will value one piece higher than the other. Since the moving knives lead to a continuous change in valuations, the Intermediate Value Theorem indicates that there will be a point t such that 2 will value the center piece at exactly 1/2, and thus they will stop the procedure there. Therefore, the procedure terminates.

ii. Player 1 moves the knives so that they always value the center piece at 1/2 and value the outer pieces at 1/2. Thus, no matter when the procedure is stopped, 1 will think each player has received a value of 1/2. Also, player 2 stops the procedure when they value the center piece at exactly 1/2, and thus they value the outer pieces at 1/2 as well, and thus believes that both players have received a value of 1/2. Thus, this procedure produces an exact division.

Adjusted Winners

To this point we've seen several two-player procedures that ensure proportionality and envy-freeness, but none that also ensure Pareto efficiency. The **Adjusted Winner Procedure** guarantees a division for two players that is both envy-free and Pareto efficient when G contains divisible, homogeneous elements.

The adjusted winner procedure proceeds as follows:

i. Suppose $G = \{g_1, g_2, \ldots, g_k\}$ is a set of k divisible, homogeneous goods, and Players 1 and 2 have value functions v_1 and v_2, respectively.

ii. Let $G_1 = \{g_i \mid v_1(g_i) > v_2(g_i)\}$, let $G_2 = \{g_i \mid v_2(g_i) > v_1(g_i)\}$, and let G_r be the remainder, namely $\{g_i \mid v_1(g_i) = v_2(g_i)\}$. Without loss of generality suppose $v_1(G_1) \geq v_2(G_2)$.

iii. Tentatively assign the goods in $G_1 \cup G_r$ to Player 1, and the goods in G_2 to Player 2, so each player gets the goods they value higher than the other player, and Player 1 gets all goods for which the values were tied.

iv. List the goods on an order $g_{i_1}, g_{i_2}, \dots g_{i_k}$ such that

$$\frac{v_1(g_{i_1})}{v_2(g_{i_1})} \geq \frac{v_1(g_{i_2})}{v_2(g_{i_2})} \geq \cdots \geq \frac{v_1(g_{i_k})}{v_2(g_{i_k})}$$

v. Then, Player 1 has been assigned goods $g_{i_1} \dots g_{i_r}$ for some r (in particular, all the goods with $\frac{v_1(g_{i_1})}{v_2(g_{i_1})} \geq 1$), and Player 2 has been assigned goods $g_{i_{r+1}} \dots g_{i_k}$.

vi. At this point, Player 1 has the equal or greater total value than Player 2. If their total values are equal, then we're done. Otherwise, Player 1 gives goods or fractions of goods to Player 2 in the order $g_{i_r}, g_{i_{r-1}}, g_{i_{r-2}}, \dots$ as necessary until both players have the same total value (based on their own value function).

In the following problem, we'll prove the fairness of the Adjusted Winner Procedure.

7. (a) **[4]** Suppose that Player 1 values g_i at least as much as Player 2 does, and Player 2 values g_j at least as much as Player 1 does. Additionally, suppose Player 1 possesses a fraction s $(0 < s \leq 1)$ of g_i, and Player 2 possesses a fraction t $(0 < t \leq 1)$ of g_j in some division. Prove that if a trade of Player 1's fraction of g_i for Player 2's fraction of g_j increases either player's total value, then the other player's total value must decrease.

   (b) **[8]** Suppose, as in (a), that Player 1 possesses a fraction s $(0 < s \leq 1)$ of g_i, and Player 2 possesses a fraction t $(0 < t \leq 1)$ of g_j in some division, and suppose $\frac{v_1(g_j)}{v_2(g_j)} \leq \frac{v_1(g_i)}{v_2(g_i)}$. Again, prove that if a trade of Player 1's fraction of g_i for Player 2's fraction of g_j increases either player's total value, then the other player's total value must decrease.

   (c) **[12]** Using previous results, prove that the Adjusted Winner Procedure yields a division that is both envy-free and Pareto efficient.

Solution to Problem 7:

(a) From the problem statement, we know $v_1(g_i) \geq v_2(g_i)$ and $v_1(g_j) \leq v_2(g_j)$. Since g_i and g_j are homogeneous, Player 1 values his share of g_i at $sv_1(g_i)$, and Player 2 vaules her share of g_j at $tv_2(g_j)$. Suppose, without loss of generality, that the trade strictly benefits Player 1. Then $tv_1(g_j) > sv_1(g_i)$. Hence, since $v_1(g_i) \geq v_2(g_i)$ and $v_1(g_j) \leq v_2(g_j)$, we know

$$tv_2(g_j) \geq tv_1(g_j) > sv_1(g_i) \geq sv_2(g_i)$$

so the value that Player 2 receives $(sv_2(g_i))$ is strictly less than the value that Player 2 had $(tv_2(g_j))$, so Player 2's total value decreases, as desired.

(b) Suppose, for contradiction, that such a trade is strictly better for one player and no worse for the other. Then both players end up with at least the same total value they had, and one player ends up with strictly greater total value. Namely, $tv_1(g_j) \geq sv_1(g_i)$ and $sv_2(g_i) \geq tv_2(g_j)$ with at least one of these inequalities strict. Multiplying the first inequality by $v_2(g_i)$ on both sides and the second by $v_1(g_i)$, we find

$$tv_1(g_j)v_2(g_i) \geq sv_1(g_i)v_2(g_i)$$

and

$$sv_2(g_i)v_1(g_i) \geq tv_2(g_j)v_1(g_i)$$

so

$$tv_1(g_j)v_2(g_i) \geq sv_1(g_i)v_2(g_i) \geq tv_2(g_j)v_1(g_i)$$

so, since at least one of the first two inequalities was strict,

$$v_1(g_j)v_2(g_i) > v_2(g_j)v_1(g_i)$$

Hence,

$$\frac{v_1(g_j)}{v_2(g_j)} > \frac{v_1(g_i)}{v_2(g_i)}$$

a contradiction.

(c) We first show that Adjusted Winner is envy-free. The procedure terminates when both players have the same total value. Let A be the set of goods that Player 1 receives, and B be the set of goods that Player 2 receives. Then we know $v_1(A) = v_2(B)$. Since $A = G - B$, this also means $v_1(B) = 1 - v_1(A) = 1 - v_2(B) = v_2(A)$. Additionally, by construction, for every element $g \in A$, we know that Player 1 values g at least as much as Player 2 does. Hence,

$$v_1(A) = \sum_{g \in A} v_1(g) \geq \sum_{g \in A} v_2(g) = v_2(A)$$

and so $v_1(A) \geq v_2(A) = v_1(B)$, so Player 1 values A at least as much as B, so Player 1 is not envious of Player 2. Similarly, $v_2(B) = v_1(A) \geq v_2(A)$, so Player 2 values B at least as much as A and Player 2 is not envious of Player 1. Hence, Adjusted Winner is envy-free.

Now, we will use 7(a), 7(b), and 3(c) to show that Adjusted Winner is also Pareto efficient. Suppose for contradiction that a division that resulted from Adjusted Winner is not efficient. Then, from 3(c), we can find g_i and g_j in G such that a trade between two players of a part of g_i for a part of g_j yields a division that dominates the given one. Suppose Player 1 posesses such a fraction s of g_i, and Player 2 posesses such a fraction t of g_j.

Since Player 1 posesses part of g_i, we know $v_1(g_i) \geq v_2(g_i)$ (otherwise Player 1 would not have been assigned any of g_i by the Adjusted Winner Procedure). Since the trade yields a dominant division, from 7(a) we know that $v_2(g_j) \not\geq v_1(g_j)$, so $v_1(g_j) > v_2(g_j)$. However, since Player 2 posesses some of g_j, we know she must have received it from Player 1 in Step vi of the Adjusted Winner Procedure. Hence, Player 2 posesses all of the goods $g_{j+1}, \dots, g_k$, so in particular since Player 1 posesses part of g_i we know $i < j$. Then from the inequalities we assumed in Part iv of the Adjusted Winner Procedure we know

$$\frac{v_1(g_i)}{v_2(g_i)} \geq \frac{v_1(g_j)}{v_2(g_j)}$$

Hence, we can apply 7(b), which implies that our originally chosen trade cannot result in a new division that dominates the original, a contradiction. Hence, a division that results from the Adjusted Winner Procedure must be efficient, as desired.

Three-player Procedures

The Selfridge-Conway Procedure, discovered in 1960, was the first discrete fair division procedure for three players that is envy-free. It's quite a bit more complex than the others we've seen. The procedure is as follows:

i. Player 1 divides G into three sets of what he believes are equal value.

ii. Let A be the set that Player 2 most prefers, let B be the next set, and let C be the set Player 2 least prefers. Player 2 partitions A into A_1 and A_2, so that A_1 has the same value to her as B, and A_2 (possibly empty) is the remainder.

iii. Player 3 chooses the set among A_1, B, and C that he most prefers.

iv. Player 2 chooses the set among the remainder that she most prefers, with the restriction that she must choose A_1 if it's available.

v. Player 1 chooses the last set from those three, leaving just A_2 to be divided (if it exists).

vi. Either Player 2 or Player 3 must have chosen A_1. Without loss of generality suppose it's Player 2 (otherwise swap the roles of players 2 and 3 in the following steps).

vii. Player 3 divides A_2 into three sets of equal value.

viii. Player 2 chooses her favorite one of these subsets.

ix. Player 1 chooses his favorite among the remaining two.

x. Player 3 chooses the final subset of A_2.

8. (a) **[8]** Prove that the Selfridge-Conway Procedure always produces an envy-free division when all elements of G are divisible (but not necessarily homogeneous).

   (b) **[8]** Note that Selfridge-Conway requires up to 5 "cuts" to be made: two to create the initial three regions, one to create A_1 and A_2, and two more to divide A_2. By relaxing our notion of fairness, we can reduce the number of cuts required.

      i. Design a new algorithm for three players, based on Selfridge-Conway and possibly others we've seen, that requires at most 3 cuts, and always produces a *proportional* division when all elements of G are divisible.

      ii. Prove that your algorithm satisfies these conditions. Namely, prove that it requires at most three cuts, and that it always produces a proportional division.

Solution to Problem 8:

(a) Suppose Player 3 picks A1. Player 2 picks B and thus Player 1 picks C. Player 2 splits up A2 into three equal sets. Player 3 cannot envy anyone as he picks first both splits. Player 2 cannot envy Player 1 since she picks before Player 1 in the first split and believes to receive an equal amount in the second split, and cannot envy Player 3 because she got an equal amount in the first split (valuing A1 and B the same), and created an equal split the second time, so she believes that she received an equal value as Player 3. Player 1 cannot envy Player 2 since he picks before Player 2 in the second split and believes to receive an equal amount in the first split, and cannot envy Player 3 because he believes to receive a third of the value, whereas the most that Player 3 can get is A1 and A2, which adds up to a third of the value.

Suppose Player 3 picks B. Player 2 picks A1 and Player 1 picks C. Player 3 splits up A2 into three equal sets. Player 3 cannot envy anyone as he picks first in the first split and believes to receive an equal amount in the second split. Player 2 cannot envy anyone as she picks first in the second split and believes to receive at least as much as the other two players in the split (as she values A1 and B equally, and A1 more than C) Player 1 cannot envy Player 3 since he believes to receive an equal amount in the first split and picks before Player 3 in the second split, and cannot envy Player 2 because he believes to receive a third of the value, whereas the most that Player 2 can get is A1 and A2, which adds up to a third of the value.

Suppose Player 3 picks C. Player 2 picks A1 and Player 1 picks B. This case is analogous to the previous one.

(b) First, we note that divide and choose is a process that requires only one cut for two players.

Player 1 splits a piece of 1/3 value. If neither Player 2 nor Player 3 value this piece at least 1/3 value, then Player 1 takes the piece and Player 2 and 3 play divide and choose on the remaining part. In this case, Player 1 will receive a piece of 1/3 value and divide and choose will guarantee Players 2 and 3 also receive pieces of 1/3 value so this is a proportional division. In addition, only two cuts are used, one by Player 1 and one in the divide and choose.

If exactly one of Player 2 and 3 value the piece at least 1/3, they take the piece, and the other two players (including Player 1) divide and choose the remaining piece. Again, either Player 2 or Player 3 will receive a piece of 1/3 value and divide and choose will guarantee the remaining players also receive pieces of 1/3 value. This also takes two cuts: the initial one by Player 1, and the one in divide and choose.

If both Player 2 and 3 value the piece at least 1/3, Player 2 splits this piece into a piece of value at least 1/3 and another piece. If Player 3 values this smaller piece at least 1/3, he takes it; otherwise, Player 2 takes it. Player 1 and the player who didn't take a piece play divide and choose on the remaining two parts. Since Player 2 splits the piece into a piece of value at least 1/3 and Player 3 takes it only if he values the split piece at least 1/3, whoever takes the split piece will have total value at least 1/3. Divide and choose guarantees the remaining players also receive total value at least 1/3. In this last case, Player 2 had to make an additional cut, so a total of three cuts are used.

Lying for Fun and Profit

To this point, we've assumed that the players are always honest, and accurately represent their true preferences. However, what happens when we remove this assumption?

9. (a) **[6]** Let $G = \{A, B, C\}$, where A, B, and C are each divisible and homogeneous. Additionally, suppose Player 1 and Player 2's true value functions are as follows:

	v_1	v_2
A	0.2	0.3
B	0.3	0.4
C	0.5	0.3

If Player 1 and Player 2 divide G using the Divide and Choose method, and Player 1 knows Player 2's value function (as well as his own), then Player 1 can guarantee himself a total value of $\frac{x}{1000}$ for many integers x by carefully selecting the initial division (knowing that Player 2 will then choose whichever half has greatest total value to her). Compute the largest such integer x.

(b) **[4]** Suppose a division is chosen by a third party that maximizes welfare. For example, using the values given in part (a), the maximal division is $\{1 : (C), 2 : (A, B)\}$, which yields a total value of 0.5 to Player 1. If Player 1 lied about his value function to the third party but Player 2 told the truth, what is the maximal true total value (using his original value function from (a)) that Player 1 could achieve?

Solution to Problem 9:

(a) Suppose Player 1 divides G into two sets $\{k_1 A, k_2 B, k_3 C\}$ and $\{(1-k_1)A, (1-k_2)B, (1-k_3)C\}$ for $0 \le k_1, k_2, k_3 \le 1$ such that Player 2 chooses the first set. Then we must have $0.3k_1+0.4k_2+0.3k_3 < 0.5$ and we wish to maximize $0.2(1-k_1)+0.3(1-k_2)+0.5(1-k_3)$. Therefore, it follows that we wish to minimize k_3, k_2, then k_1 in that order. Thus, it follows that $k_1 = 1$, $k_3 = 0$, and we want to make k_2 as close to 0.5 as possible.

Therefore, we can guarantee a total value for Player 1 that is arbitrarily close to $0.2(1-1)+0.3(1-0.5)+0.5(1-0) = 0.65$. The greatest such integer x such that $\frac{x}{1000} < 0.65$ is $x = 649$ so 649 is the answer.

(b) The division that maximizes welfare will assign each good to whichever player values it more. Player 1 can't win all three goods, but he can win B and C (the two he values most highly) by, for example, pretending each has value 0.5 to him. Then Player 1 will receive B and C for a total value of $\boxed{0.8}$.

Note that in general the lying player can guarantee that he wins all but one good.