1. Given that x and y are nonnegative integers, compute the number of pairs (x, y) such that $5x + y = 20$.

   Answer: 5

   Solution: For $x = 0, 1, 2, 3, 4$, there is a valid y, giving the answer $\boxed{5}$.

2. $f(x) = x^2 + bx + c$ is a function with the property that the x-coordinate of the vertex is equal to the positive difference of the two roots of $f(x)$. Given that $c = 48$, compute b.

   Answer: −16

   Solution 1: The x-coordinate of the vertex can be written as $\frac{-b}{2a}$, and the difference of the two roots can be written as $\frac{\sqrt{b^2-4ac}}{a}$ by using the quadratic formula. Equating these two values and plugging in for a and c, we have

$$\begin{aligned}
\frac{-b}{2a} &= \frac{\sqrt{b^2-4ac}}{a} \\
\frac{-b}{2} &= \sqrt{b^2-4\cdot 48} \\
\frac{b^2}{4} &= b^2 - 4\cdot 48 \\
4\cdot 48 &= \frac{3}{4}b^2 \\
16\cdot 16 &= b^2 \\
b &= \pm 16
\end{aligned}$$

   Because the x-coordinate must be positive, $\frac{-b}{2a}$ must be positive, so $b = \boxed{-16}$.

   Solution 2: Note that the x-coordinate of the vertex is equal to the average of the two roots of $f(x)$. Letting the two roots be r_1 and r_2, where r_2 is larger, we may then write

$$\begin{aligned}
\frac{r_1+r_2}{2} &= r_2 - r_1 \\
r_1 + r_2 &= 2r_2 - 2r_1 \\
3r_1 &= r_2
\end{aligned}$$

   Hence, $r_1 \cdot r_2 = 3r_1^2 = 48$, and $r_1 = 4$. We can then compute that $r_2 = 12$. Finally, $b = -(r_1 + r_2) = -(4 + 12) = \boxed{-16}$.

3. Suppose we have a function $f(x)$ such that $f(x)^2 = f(x-5)f(x+5)$ for all integers x. Given that $f(1) = 1$ and $f(16) = 8$, what is $f(2016)$?

   Answer: $\mathbf{2^{403}}$

   Solution: First, we can easily compute that $f(6) = 2$ and $f(11) = 4$. Note that

$$\frac{f(x)}{f(x-5)} = \frac{f(x+5)}{f(x)} = \frac{f(6)}{f(1)} = 2,$$

   hence $\frac{f(x+5)}{f(x)} = 2$ holds for all $x \equiv 1 \pmod 5$. We claim that $f(5n+1) = 2^n$ for all non-negative integers n and proceed by induction.

For the base case, when $n = 0$, we have that $f(1) = 1 = 2^0$; hence, the base case holds. For the inductive step, assume that $f(5n+1) = 2^n$. We wish to show that $f(5n+6) = 2^{n+1}$. Notice that

$$f(5n+6) = f(5n+1) \cdot \frac{f(5n+6)}{f(5n+1)} = 2^n \cdot 2 = 2^{n+1},$$

completing the inductive step. Thus, we have shown that $f(5n+1) = 2^n$ for all non-negative integers n. Finally, $f(2016) = f(403 \cdot 5 + 1) = \boxed{2^{403}}$, as desired.

4. Suppose that we have the following set of equations

$$\log_2 x + \log_3 x + \log_4 x = 20$$
$$\log_4 y + \log_9 y + \log_{16} y = 16$$

Compute $\log_x y$.

Answer: $\frac{8}{5}$ or 1.6

Solution 1: We rewrite the first equation as

$$\begin{aligned}
20 &= \log_2 x + \log_3 x + \log_4 x \\
20 &= \frac{\log x}{\log 2} + \frac{\log x}{\log 3} + \frac{\log x}{\log 4} \\
20 &= \log x \cdot (\log_2 10 + \log_3 10 + \log_4 10)
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
16 &= \log y \cdot (\log_4 10 + \log_9 10 + \log_{16} 10) \\
16 &= \log y \cdot (\log_2 \sqrt{10} + \log_3 \sqrt{10} + \log_4 \sqrt{10}) \\
16 &= \frac{1}{2} \log y \cdot (\log_2 10 + \log_3 10 + \log_4 10) \\
32 &= \log y \cdot (\log_2 10 + \log_3 10 + \log_4 10)
\end{aligned}$$

Dividing both equations by each other, we obtain

$$\begin{aligned}
\frac{32}{20} &= \frac{\log y}{\log x} \frac{\log_2 10 + \log_3 10 + \log_4 10}{\log_2 10 + \log_3 10 + \log_4 10} \\
\boxed{\frac{8}{5}} &= \log_x y
\end{aligned}$$

Solution 2: We use a similar technique in the first solution, but use base x rather than base 10 to convert the second equation. This gives us

$$\begin{aligned}
16 &= \frac{\log_x y}{\log_x 4} + \frac{\log_x y}{\log_x 9} + \frac{\log_x y}{\log_x 16} \\
16 &= \log_x y \left(\frac{1}{2\log_x 2} + \frac{1}{2\log_x 3} + \frac{1}{2\log_x 4} \right) \\
16 &= \frac{1}{2} (\log_2 x + \log_3 x + \log_4 x) \log_x y
\end{aligned}$$

From the first equation, $\log_2 x + \log_3 x + \log_4 x = 20$, so $16 = \frac{1}{2} \cdot 20 \cdot \log_x y$, which gives us the desired $\log_x y = \boxed{\frac{8}{5}}$.

5. Let $\{a_n\}$ be the arithmetic sequence defined as $a_n = 2(n-1)+6$ for all $n \geq 1$. Compute

$$\sum_{i=1}^{\infty} \frac{1}{a_i a_{i+2}}$$

Answer: $\mathbf{\frac{7}{96}}$

Solution: We rewrite the sum as follows:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{a_i a_{i+2}} &= \sum_{i=1}^{\infty} \frac{1}{(a_i)(a_i + 2\cdot 2)} \\ &= \sum_{i=1}^{\infty} \frac{1}{2\cdot 2}\left(\frac{1}{a_i} - \frac{1}{a_i+4}\right) \\ &= \sum_{i=1, i \text{ odd}}^{\infty} \frac{1}{4}\left(\frac{1}{a_i} - \frac{1}{a_{i+2}}\right) + \sum_{i=1, i \text{ even}}^{\infty} \frac{1}{4}\left(\frac{1}{a_i} - \frac{1}{a_{i+2}}\right)\end{aligned}$$

The first sum telescopes to $\frac{1}{4\cdot 6}$, while the second sum telescopes to $\frac{1}{4\cdot 8}$. Hence, the total sum is

$$\frac{1}{4\cdot 6} + \frac{1}{4\cdot 8} = \frac{4+3}{4\cdot 24} = \boxed{\frac{7}{96}}$$

6. Let a, b, c, d, e, f be non-negative real numbers. Suppose that $a+b+c+d+e+f = 1$ and $ad + be + cf \geq \frac{1}{18}$. Find the maximum value of $ab + bc + cd + de + ef + fa$.

Answer: $\mathbf{\frac{7}{36}}$

Solution: We partially factor

$$ab + bc + cd + de + ef + fa = a(b+f) + c(b+d) + e(d+f).$$

Notice that the terms in the parenthesis are very close to $(b+d+f)$, except each part is missing one term. We can add $ad + cf + eb$ to this expression to get

$$\begin{aligned}&a(b+f) + c(b+d)e(d+f) + ad + cf + eb \\ &= a(b+d+f) + c(b+d+f) + e(b+d+f) \\ &= (a+c+e)(b+d+f)\end{aligned}$$

Let $x = a+c+e$ and $y = b+d+f$. We have $x+y=1$, so AM-GM gives us $\frac{x+y}{2} \leq \sqrt{xy}$, hence $(a+c+e)(b+d+f) \leq \frac{1}{4}$. Subtracting away $ad+be+cf \geq \frac{1}{18}$ gives us

$$ab + bc + cd + de + ef + fa \leq \frac{1}{4} - \frac{1}{18} = \boxed{\frac{7}{36}}$$

where equality holds when $a+c+e = b+d+f = \frac{1}{2}$ and $ad+be+cf = \frac{1}{18}$. We can verify that equality holds when $a = e = \frac{1}{9}, b = d = \frac{1}{4}, c = \frac{5}{18}, f = 0$.

7. Let f be a continuous real-valued function defined on the positive real numbers. Determine all f such that for all positive real x, y we have $f(xy) = xf(y) + yf(x)$ and $f(2016) = 1$.

Answer: $\mathbf{\frac{1}{2016} x \log_{2016} x}$

Solution: For any positive real a and any positive integer n, we claim that $f(a^n) = na^{n-1}f(a)$. We prove this by induction. For the base case, $n = 1$, we clearly have $f(a^1) = f(a) = 1 \cdot a^0 f(a)$, so the base case holds. For the inductive step, suppose that $f(a^k) = ka^{k-1}f(a)$. We wish to show that $f(a^{k+1}) = (k+1)a^k f(a)$. Notice that

$$f(a^{k+1}) = f(a^k \cdot a) = a^k f(a) + af(a^k) = a^k f(a) + ka^k f(a) = (k+1)a^k f(a),$$

which completes the inductive step. Therefore, $f(a^n) = na^{n-1}f(a)$ for all positive real a and positive integer n.

Next, notice that

$$f(1) = f(1 \cdot 1) = 1f(1) + 1f(1) = 2f(1).$$

This implies that $f(1) = 0$. Now for any positive a, we can write

$$0 = f(1) = f\left(a \cdot \frac{1}{a}\right) = af\left(\frac{1}{a}\right) + \frac{1}{a}f(a),$$

hence $f(\frac{1}{a}) = -\frac{1}{a^2}f(a)$. Therefore,

$$f(a^{-n}) = f\left(\left(\frac{1}{a}\right)^n\right) = n\left(\frac{1}{a}\right)^{n-1} f\left(\frac{1}{a}\right) = na^{-n+1}\left(-\frac{1}{a^2}f(a)\right) = -na^{-n-1}f(a).$$

Therefore, we have extended $f(a^n) = na^{n-1}f(a)$ to hold for all positive real a and all integer n.

Now notice that we can write

$$f(a) = f\left(\left(a^{\frac{1}{n}}\right)^n\right) = n\left(a^{\frac{1}{n}}\right)^{n-1} f\left(a^{\frac{1}{n}}\right) = na^{1-\frac{1}{n}} f\left(a^{\frac{1}{n}}\right).$$

Isolating $f(a^{\frac{1}{n}})$ gives us $f(a^{\frac{1}{n}}) = \frac{1}{n}a^{\frac{1}{n}-1}f(a)$. Hence, for all rational numbers $q = \frac{m}{n}$, where m and n are integers, and all positive a, we have

$$\begin{aligned} f(a^q) &= f\left(a^{\frac{m}{n}}\right) = f\left(\left(a^{\frac{1}{n}}\right)^m\right) = m\left(a^{\frac{1}{n}}\right)^{m-1} f\left(a^{\frac{1}{n}}\right) \\ &= ma^{\frac{m}{n}-\frac{1}{n}}\left(\frac{1}{n}a^{\frac{1}{n}-1}f(a)\right) = \frac{m}{n}a^{\frac{m}{n}-1}f(a) \\ &= qa^{q-1}f(a). \end{aligned}$$

Now, because f is continuous, for all real numbers r and all positive real numbers a, we have $f(a^r) = ra^{r-1}f(a)$. Therefore, for all positive real x, we have

$$f(x) = f(2016^{\log_{2016} x}) = (\log_{2016} x)2016^{\log_{2016} x - 1} f(2016) = x\log_{2016} x \cdot \frac{1}{2016} = \boxed{\frac{1}{2016}x\log_{2016} x}.$$

8. Find the maximum of the following expression:

$$21\cos\theta + 18\sin\theta\sin\phi + 14\sin\theta\cos\phi.$$

Answer: 31

Solution: Using the Cauchy-Schwarz inequality,

$$\begin{aligned}(21\cos\theta + 18\sin\theta\sin\phi + 14\sin\theta\cos\phi)^2 &\le (21^2+18^2+14^2)(\cos^2\theta + \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi)\\ &= 961(\cos^2\theta + \sin^2\theta)\\ &= 961\end{aligned}$$

Therefore, $|21\cos\theta + 18\sin\theta\sin\phi + 14\sin\theta\cos\phi| \le 31$. We note that equality holds when

$$\frac{\cos\theta}{21} = \frac{\sin\theta\sin\phi}{18} = \frac{\sin\theta\cos\phi}{14}.$$

This is true when $\phi = \arctan\frac{9}{7}, \theta = \arctan\frac{6}{7\sin\phi}$, so the inequality must be tight. Furthermore, the transformation $\theta \mapsto \pi - \theta, \phi \mapsto \pi - \phi$ will flip the sign of the given expression, so the maximum $\boxed{31}$ is indeed attained for some values of θ, ϕ.

9. a, b, c, d satisfy the following system of equations

$$\begin{aligned}ab + c + d &= 13\\ bc + d + a &= 27\\ cd + a + b &= 30\\ da + b + c &= 17\end{aligned}$$

Compute the value of $a + b + c + d$.

Answer: 70

Solution: Let $s = a + b + c + d$. Subtracting s from each side of the equations gives us

$$\begin{aligned}ab - a - b &= 13 - s\\ bc - b - c &= 27 - s\\ cd - c - d &= 30 - s\\ da - d - a &= 17 - s\end{aligned}$$

Adding one to each side of the equations and applying Simon's Favorite Factoring Trick gives us

$$\begin{aligned}(a-1)(b-1) &= 14 - s\\ (b-1)(c-1) &= 28 - s\\ (c-1)(d-1) &= 31 - s\\ (d-1)(a-1) &= 18 - s\end{aligned}$$

Multiplying the first and third equations and the second and fourth equations allows us to write the following equality:

$$(a-1)(b-1)(c-1)(d-1) = (14-s)(31-s) = (28-s)(18-s)$$

Expanding and solving for s gives us $s = \boxed{70}$.

10. Define a sequence of numbers $a_{n+1} = \frac{(2+\sqrt{3})a_n + 1}{(2+\sqrt{3}) - a_n}$ for $n > 0$, and suppose that $a_1 = 2$. What is a_{2016}?

Answer: $\mathbf{\frac{6+5\sqrt{3}}{13}}$

After dividing the numerator and the denominator by $2+\sqrt{3}$, we have the recurrence

$$a_{n+1} = \frac{(2-\sqrt{3})+a_n}{1-(2-\sqrt{3})a_n}$$

Let $\theta_n = \tan^{-1}(a_n)$. Note that we can rewrite the recurrence as $\tan\theta_{n+1} = \frac{\tan\theta_n + \tan(\pi/24)}{1-\tan\theta_n \tan(\pi/24)}$, since $2-\sqrt{3} = \tan(\pi/24)$. (We can notice that $\tan(\pi/24) = 2-\sqrt{3}$ by using the half angle formula on $\tan(\pi/12) = 1/\sqrt{3}$, which has a $\sqrt{3}$ term in it.) We then see that this recurrence has a period of 24, and hence $a_1 = a_{2017} = 1$. Hence, we can solve for a_{2016} by stepping backwards through the recurrence once:

$$\begin{aligned}
a_{2017} &= \frac{(2+\sqrt{3})a_{2016}+1}{(2+\sqrt{3})-a_{2016}} \\
2(2+\sqrt{3}-a_{2016}) &= (2+\sqrt{3})a_{2016}+1 \\
4+2\sqrt{3}-2a_{2016} &= (2+\sqrt{3})a_{2016}+1 \\
3+2\sqrt{3} &= (4+\sqrt{3})a_{2016} \\
a_{2016} &= \frac{3+2\sqrt{3}}{4+\sqrt{3}}
\end{aligned}$$

Rationalizing the denominator gives us $a_{2016} = \boxed{\frac{6+5\sqrt{3}}{13}}$.