1. Evaluate
$$\int_{-2}^{2} \left(x^3 + 2x + 1\right) \, dx.$$

Answer: 4

Solution 1: We directly compute that

$$\int_{-2}^{2} \left(x^3 + 2x + 1\right) \, dx = \left[\frac{x^4}{4} + x^2 + x\right]_{-2}^{2} = (4 + 4 + 2) - (4 + 4 - 2) = \boxed{4}.$$

Solution 2: Observe that $x^3 + 2x$ is an odd function, so we get 0 when we integrate it over the symmetric interval $[-2, 2]$. Therefore, we see that

$$\int_{-2}^{2} \left(x^3 + 2x + 1\right) \, dx = \int_{-2}^{2} 1 \, dx = [x]_{-2}^{2} = 2 - (-2) = \boxed{4}.$$

2. Find
$$\lim_{x \to 0} \frac{\ln(1 + x + x^3) - x}{x^2}.$$

Answer: $-\frac{1}{2}$ or -0.5

Solution: We apply L'Hôpital's rule and obtain

$$\begin{aligned}
\lim_{x \to 0} \frac{\ln(1 + x + x^3) - x}{x^2} &= \lim_{x \to 0} \frac{(\ln(1 + x + x^3) - x)'}{(x^2)'} \\
&= \lim_{x \to 0} \left(\frac{1}{2x}\right)\left(\frac{1 + 3x^2}{1 + x + x^3} - 1\right) \\
&= \lim_{x \to 0} \left(\frac{1}{2x}\right)\left(\frac{-x + 3x^2 - x^3}{1 + x + x^3}\right) \\
&= \lim_{x \to 0} \left(\frac{-1 + 3x - x^2}{2(1 + x + x^3)}\right) \\
&= \boxed{-\frac{1}{2}}.
\end{aligned}$$

3. Find the largest possible value for the slope of a tangent line to the curve $f(x) = \frac{1}{3+x^2}$.

Answer: $\frac{1}{8}$ or 0.125

Solution: The slopes of the tangent lines of $f(x)$ are exactly the values of $f'(x)$. We want to maximize $f'(x)$. So let's take some derivatives! We have

$$\begin{aligned}
f'(x) &= \frac{-2x}{(3 + x^2)^2}, \\
f''(x) &= \frac{-2(3 + x^2)^2 + 8x^2(3 + x^2)}{(3 + x^2)^4}.
\end{aligned}$$

In order to maximize $f'(x)$, we solve $f''(x) = 0$ and look for critical points. To do this, we only need to consider the numerator:

$$\begin{aligned} -2(3+x^2)^2 + 8x^2(3+x^2) &= 0 \\ (-6 - 2x^2 + 8x^2)(3+x^2) &= 0 \\ -6 + 6x^2 &= 0 \\ x &= \pm 1. \end{aligned}$$

We see that $f'(-1) = \frac{1}{8}$ and $f'(1) = -\frac{1}{8}$, so the largest possible slope is therefore $\boxed{\frac{1}{8}}$.

4. An empty, inverted circular cone has a radius of 5 meters and a height of 20 meters. At time $t = 0$ seconds, the cone is empty, and at time $t \geq 0$ we fill the cone with water at a rate of $4t^2$ cubic meters per second. Compute the rate of change of the height of water with respect to time, at the point when the water reaches a height of 10 meters.

   Answer: $\frac{4}{\sqrt[3]{\pi}}$

   Solution: The volume of water at time t is

$$\int_0^t 4s^2\, ds = \frac{4}{3}t^3$$

Suppose the height of the water at time t is $h(t)$. Then, the water fills a cone of height $h(t)$ and radius $r(t) = \frac{1}{4}h(t)$. Hence, the volume of the cone of water is also

$$\frac{1}{3}h(t)(\pi r^2(t)) = \frac{\pi}{48}h^3(t).$$

Equating the two equations for volume, we have

$$\begin{aligned} \frac{4}{3}t^3 &= \frac{\pi}{48}h^3(t) \\ 64t^3 &= \pi h^3(t) \\ \frac{4}{\sqrt[3]{\pi}}t &= h(t) \end{aligned}$$

Differentiating $h(t)$ with respect to t, we conclude that $h'(t) = \boxed{\frac{4}{\sqrt[3]{\pi}}}$.

5. Compute

$$\int_0^{\frac{\pi}{2}} \sin(2016x)\cos(2015x)\; dx.$$

   Answer: $\frac{2016}{4031}$

   Solution: We use the identity

$$\sin(x)\cos(y) = \frac{\sin(x+y) + \sin(x-y)}{2}$$

Substituting the identity into the integral, we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin(2016x)\cos(2015x)\ dx &= \int_0^{\frac{\pi}{2}} \frac{\sin(4031x)+\sin(x)}{2}\ dx \\ &= \left[-\frac{\cos(4031x)}{4031\cdot 2} - \frac{\cos(x)}{2}\right]_0^{\frac{\pi}{2}} \\ &= 0 - \left(-\frac{1}{4031\cdot 2} - \frac{1}{2}\right) \\ &= \boxed{\frac{2016}{4031}}.\end{aligned}$$

6. Let $f(x)$ be a function defined for $x > 1$ such that $f''(x) = \frac{x}{\sqrt{x^2-1}}$ and $f'(2) = \sqrt{3}$. Compute the length of the graph of $f(x)$ on the domain $x \in (1, 2]$.

Answer: $\mathbf{\frac{3}{2}}$ **or 1.5**

Solution: Taking the integral of $f''(x)$, we see that $f'(x) = \sqrt{x^2-1} + C$. Plugging in $x = 2$, we obtain

$$\sqrt{2^2-1} + C = \sqrt{3} \implies C = 0.$$

Thus, the length of the graph of $f(x)$ on the domain $x \in (1, 2]$ is

$$\int_1^2 \sqrt{1+(f'(x))^2}\ dx = \int_1^2 \sqrt{1+(x^2-1)}\ dx = \int_1^2 x\ dx = \boxed{\frac{3}{2}}.$$

7. Let the function $f : [1, \infty) \to \mathbb{R}$ be defined as $f(x) = x^{2\ln(x)}$. Compute

$$\int_1^{\sqrt{e}} \left(f(x) + f^{-1}(x)\right)\ dx.$$

Answer: $\boldsymbol{e - 1}$

Solution: Note that f maps $[1, \sqrt{e}]$ to itself monotonically and thus bijectively, hence the substitution of $f(x)$ for x gives

$$\int_1^{\sqrt{e}} \left(f(x) + f^{-1}(x)\right)\ dx = \int_1^{\sqrt{e}} f(x)\,dx + \int_1^{\sqrt{e}} f^{-1}(x)\,dx = \int_1^{\sqrt{e}} f(x)\,dx + \int_1^{\sqrt{e}} x\,d(f(x)).$$

Reversing integration by parts gives

$$\begin{aligned}\int_1^{\sqrt{e}} f(x)\,dx + \int_1^{\sqrt{e}} x\,d(f(x)) &= \int_1^{\sqrt{e}} \left(f(x)\,dx + x\,d(f(x))\right) \\ &= \int_1^{\sqrt{e}} d(xf(x)) \\ &= (xf(x))|_1^{\sqrt{e}} = e - 1,\end{aligned}$$

since $f(x) = x$ at the endpoints $x = 1$ and $x = \sqrt{e}$.

Alternately, one may draw a graph of f in the interval $[1, \sqrt{e}]$ and deduce the result by inspection.

8. Calculate $f(3)$, given that $f(x) = x^3 + f'(-1)x^2 + f''(1)x + f'(-1)f(-1)$.

   Answer: 198

   Solution: Differentiating, we compute

$$\begin{aligned} f'(x) &= 3x^2 + 2f'(-1)x + f''(1) \\ f''(x) &= 6x + 2f'(-1). \end{aligned}$$

Plugging in $x = -1$ into the first equation and $x = 1$ into the second, we obtain a system of equations in $f'(-1)$ and $f''(1)$:

$$\begin{aligned} f'(-1) &= 3 - 2f'(-1) + f''(1), \\ f''(1) &= 6 + 2f'(-1). \end{aligned}$$

Denoting $f'(-1)$ and $f''(1)$ by a and b respectively, we have

$$\begin{aligned} a &= 3 - 2a + b \\ b &= 6 + 2a. \end{aligned}$$

Plugging the second equation into the first, we get

$$a = 3 - 2a + (6 + 2a) = 9.$$

Thus, $b = 6 + 2a = 24$, so

$$f(x) = x^3 + ax^2 + bx + af(-1) = x^3 + 9x^2 + 24x + 9f(-1).$$

Finally, plugging $x = -1$ into this equation, we get

$$\begin{aligned} f(-1) &= -1 + 9 - 24 + 9f(-1) \\ -8f(-1) &= -16 \\ f(-1) &= -2 \end{aligned}$$

We conclude that

$$\begin{aligned} f(x) &= x^3 + 9x^2 + 24x + 18 \\ f(3) &= 3^3 + 9 \cdot 3^2 + 24 \cdot 3 + 18 \\ &= 27 + 81 + 72 + 18 = \boxed{198} \end{aligned}$$

9. Compute

$$\int_1^e \frac{\ln(x)}{(1 + \ln(x))^2}\, dx.$$

   Answer: $\frac{e}{2} - 1$ or $\frac{e-2}{2}$

Solution 1: Set $u = 1 + \ln(x)$. Then $e^{u-1} = x$ and $e^{u-1}\,du = dx$, so we have

$$\int_1^e \frac{\ln(x)}{(1+\ln(x))^2}\,dx = \int_0^2 \frac{(u-1)e^{u-1}}{u^2}\,du = \int_0^2 \left(\frac{1}{u} - \frac{1}{u^2}\right) e^{u-1}\,du.$$

Now integrate only the first term by parts, and the remaining integrals cancel:

$$\left[\frac{e^{u-1}}{u}\right]_1^2 - \int_0^2 \frac{-1}{u^2} e^{u-1}\,du - \int_0^2 \frac{1}{u^2} e^{u-1}\,du = \boxed{\frac{e}{2} - 1}.$$

Solution 2: We see that the numerator of the integral is $\ln x\,dx$. This suggests that we should do a substitution like $u = x\ln x - x$ to use this numerator. But we aren't quite ready for that – first, we want to use a substitution to set this up.

Start with $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\,dx$, and hence $-\frac{1}{u^2}\,du = dx$. The integral then becomes

$$\int_1^{1/e} \frac{\ln u}{(1-\ln u)^2} \frac{1}{u^2}\,du = \int_1^{1/e} \frac{\ln u}{(u\ln u - u)^2}\,du.$$

Next, substitute $w = u\ln u - u$. Then $dw = \ln u\,du$, so the integral becomes

$$\int_{-1}^{-2/e} \frac{dw}{w^2} = \left[-\frac{1}{w}\right]_{-1}^{-2/e} = \boxed{\frac{e}{2} - 1}.$$

10. For $x \geq 0$, let $\mathcal{R}$ be the region in the plane bounded by the graphs of the line $\ell : y = 4x$ and $y = x^3$. Let V be the volume of the solid formed by revolving $\mathcal{R}$ about line ℓ. Then V can be expressed in the form $\frac{\pi\cdot 2^a}{b\sqrt{c}}$, where a, b, and c are positive integers, b is odd, and c is not divisible by the square of a prime. Compute $a + b + c$.

Answer: 132

Solution: Note that ℓ intersects the graph of $y = x^3$ at the origin and the point $(2, 8)$ in the first quadrant. Fix d between 0 and 2. Let the line $x = d$ intersect line ℓ at Q and let the line $x = d$ intersect the graph of $y = x^3$ at R. Then $Q = (d, 4d)$ and $R = (d, d^3)$. Now drop a perpendicular from R to line ℓ, and let P be the foot of this perpendicular. Then

$$\sin\angle PQR = \frac{PR}{QR} = \frac{PR}{4d - d^3} = \frac{1}{\sqrt{17}}.$$

We compute V by integrating from 0 to 2 using the disk method. Note that the differential disk has a radius of $PR = \frac{4d-d^3}{\sqrt{17}}$ and a height of $\sqrt{17}\,dx$. So we integrate from 0 to 2 the expression

$\pi\left(\frac{4x-x^3}{\sqrt{17}}\right)^2\sqrt{17}\,dx$:

$$\begin{aligned} V=\int_0^2 \pi\left(\frac{4x-x^3}{\sqrt{17}}\right)^2\sqrt{17}\,dx &= \frac{\pi}{\sqrt{17}}\int_0^2\left(x^6-8x^4+16x^2\right)\,dx \\ &= \frac{\pi}{\sqrt{17}}\left(\frac{1}{7}x^7-\frac{8}{5}x^5+\frac{16}{3}x^3\right)\Bigg|_0^2 \\ &= \frac{\pi}{\sqrt{17}}\left(\frac{128}{7}-\frac{256}{5}+\frac{128}{3}\right) \\ &= \frac{128\pi}{\sqrt{17}}\left(\frac{1}{7}-\frac{2}{5}+\frac{1}{3}\right) \\ &= \frac{128\pi}{\sqrt{17}}\left(\frac{8}{105}\right) \\ &= \frac{\pi\cdot 2^{10}}{105\sqrt{17}}. \end{aligned}$$

Thus $a=10, b=105$, and $c=17$, and the answer is $10+105+17=\boxed{132}$.