1. Let $f(x) = x + x(\log x)^2$. Find x such that $xf'(x) = 2f(x)$.

   Answer: $\boldsymbol{e}$

   Solution: We note that $f'(x) = (1 + \log x)^2$, so the equation reduces to

$$(1 + \log x)^2 = 2(1 + (\log x)^2)$$

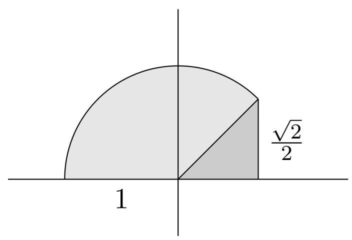
   after canceling factors of x. Factoring the difference as a square shows that the only solution occurs at $\log x = 1$, hence $x = \boxed{e}$.

2. Compute

$$\int_{-1}^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}\,dx.$$

   Answer: $\boldsymbol{\frac{3\pi}{8} + \frac{1}{4}}$ **or** $\boldsymbol{\frac{3\pi+2}{8}}$

   Solution 1: Observe that $f(x) = \sqrt{1-x^2}$ is the equation of the unit circle, centered at the origin. We need to find the area under this circular arc. We can decompose this region into two pieces: 3/8 of a circle and a right triangle, as in the diagram below:



   Therefore, the area of this region is $\boxed{\dfrac{3\pi}{8} + \dfrac{1}{4}}$.

   Solution 2: We apply the substitution $x = \sin u$. Then $dx = \cos u\,du$, and our integral becomes

$$\int_{-1}^{\frac{\sqrt{2}}{2}} \sqrt{1-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \sqrt{1-\sin^2 u}\cos u\,du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 u\,du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos(2u)+1}{2}\,du$$
$$= \left[\frac{1}{4}\sin(2u) + \frac{u}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}} = \left(\frac{1}{4} + \frac{\pi}{8}\right) - \left(0 - \frac{\pi}{4}\right) = \boxed{\frac{3\pi}{8} + \frac{1}{4}}.$$

3. An axis-aligned rectangle has vertices at $(0,0)$ and $(2,2016)$. Let $f(x,y)$ be the maximum possible area of a circle with center at (x,y) contained entirely within the rectangle. Compute the expected value of f over the rectangle.

   Answer: $\boldsymbol{\dfrac{2015\pi}{6048}}$

   Solution: Consider the right triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$. The integral of f over this region is

$$\int_0^1 \int_0^x \pi y^2\,dy\,dx = \frac{\pi}{12}.$$

Multiplying this by 8 due to symmetry gives us $\frac{2\pi}{3}$. Now, the integral of f over the axis-aligned rectangle with top vertices at $(0,1)$ and $(1,2015)$ and bottom vertices on the axis is

$$\int_0^1 \int_1^{2015} \pi x^2 \, dy \, dx = \frac{2014\pi}{3}.$$

Multiplying this by 2 due to symmetry gives us $\frac{4028\pi}{3}$. The expected value of f is therefore

$$\frac{4030\pi}{3 \cdot 2 \cdot 2016} = \boxed{\frac{2015\pi}{6048}}.$$