1. A class of six students has to split into two indistinguishable teams of three people. Compute the number of distinct team arrangements that can result.

   Answer: 10

   Solution: Consider one specific person of the six students. There are $\binom{5}{2}$ to choose the 2 partners on his team. Hence, there are $\binom{5}{2} = \boxed{10}$ distinct teams.

2. What is the probability that a randomly chosen factor of 2016 is a perfect square?

   Answer: $\frac{1}{6}$

   Solution: Note that the prime factorization of 2016 is $2016 = 2^5 3^2 7$. A factor of 2016, x is a square number only if each prime factor of x has an even multiplicity (i.e. each prime factor is multiplied an even number of times.) There are 3 ways to select powers of 2 $(2^0, 2^2, 2^4)$, 2 ways to select powers of 3 $(3^0, 3^2)$, and 1 way to select powers of 7 (7^0), for a total of $3 * 2 = 6$ square factors. Next, there are a total of $(5+1)(2+1)(1+1) = 36$ factors of 2016. Hence, the probability that a random factor of 2016 is a perfect square is $\frac{6}{36} = \boxed{\frac{1}{6}}$.

3. Compute the remainder when:

   $$5\underbrace{6666\ldots6666}_{\text{2016 sixes}}5$$

   is divided by 17.

   Answer: 4

   Solution: A neat fact: note that

   $$55 \equiv 565 \equiv 5665 \equiv 56665 \equiv \ldots \pmod{17} .$$

   This is easier to recognize when written in the form

   $$5 \equiv 56 \equiv 566 \equiv 5666 \equiv 56666 \equiv \ldots \pmod{17} ,$$

   since $10n + 6 \equiv n \pmod{17}$ when $n \equiv 5 \pmod{17}$. Then, since

   $$5\underbrace{6666\ldots6666}_{\text{2016 sixes}} \equiv 5 \pmod{17} ,$$

   clearly,

   $$5\underbrace{6666\ldots6666}_{\text{2016 sixes}}5 \equiv \boxed{4} \pmod{17} .$$

4. At an M&M factory, two types of M&Ms are produced, red and blue. The M&Ms are transported individually on a conveyor belt. Anna is watching the conveyor belt, and has determined that four out of every five red M&Ms are followed by a blue one, while one out of every six blue M&Ms is followed by a red one. What proportion of the M&Ms are red?

   Answer: $\frac{5}{29}$

   Solution: Suppose that the fraction of red M&Ms is f_r and the fraction of blue M&Ms is f_b. Then the probability that Anna sees a red one emerge next is f_r. This probability can also be expressed using conditional probabilities as $f_r = P(\text{blue})P(\text{red after blue}) + P(\text{red})P(\text{red after red})$, or $f_r = f_b(1/6) + f_r(1/5)$. Likewise, we find $f_b = f_r(4/5) + f_b(5/6)$. Solving the system of equations gives $f_b = \frac{24}{29}$ and $f_r = \boxed{\frac{5}{29}}$.

5. Three cards are chosen from a standard deck of 52 without replacing them. Given that the ace of spades was chosen, what is the expected number of aces chosen?

   Answer: $\mathbf{\frac{19}{17}}$ **or** $\mathbf{1\frac{2}{17}}$

   Solution: Let X_i be 1 if an ace was chosen on the ith draw and 0 otherwise. We wish to calculate

$$E\left[\sum_{i=1}^{3} X_i \,\middle|\, \text{the ace of spades chosen}\right].$$

   We know that $E\left[\sum_{i=1}^{3} X_i\right] = \frac{3}{13}$. If the event that the ace of spades is chosen is denoted A,

$$\begin{aligned} E\left[\sum_{i=1}^{3} X_i\right] &= E\left[\sum_{i=1}^{3} X_i \,\middle|\, A\right] P(A) + E\left[\sum_{i=1}^{3} X_i \,\middle|\, A^c\right] P(A^c) \\ &= E\left[\sum_{i=1}^{3} X_i \,\middle|\, A\right] P(A) + \sum_{i=1}^{3} E\left[X_i \mid A^c\right] P(A^c). \end{aligned}$$

   Thus,

$$\frac{3}{13} = E\left[\sum_{i=1}^{3} X_i \,\middle|\, A\right]\left(\frac{3}{52}\right) + 3\left(\frac{3}{51}\right)\left(\frac{49}{52}\right) \implies E\left[\sum_{i=1}^{3} X_i \,\middle|\, A\right] = \boxed{\frac{19}{17}}.$$

6. Moor decides that he needs a new email address, and forms the address by taking some permutation of the 12 letters $MMMOOOOOORRR$. How many permutations of the letters will contain $MOOR$ in this exact order at least once?

   Answer: 3601

   Solution: We proceed using the Principal of Inclusion-Exclusion:

   (a) To compute the number of permutations containing one instance of $MOOR$ with overcounting, we treat $MOOR$ as one character. Then, we must arrange 9 characters and divide by $4!\,2!\,2!$ since there are 4 O's, 2 M's, and 2 R's. The total number of permutations is hence $\frac{9!}{4!\,2!\,2!} = 3780$.

   (b) To compute the number of permutations containing two instances of $MOOR$ with overcounting, we must arrange 6 characters (again, counting $MOOR$ as one single character). Then, we must divide by $2!\,2!$ since there are 2 $MOOR$'s and 2 O's. The total number of permutations is hence $\frac{6!}{2!\,2!} = 180$.

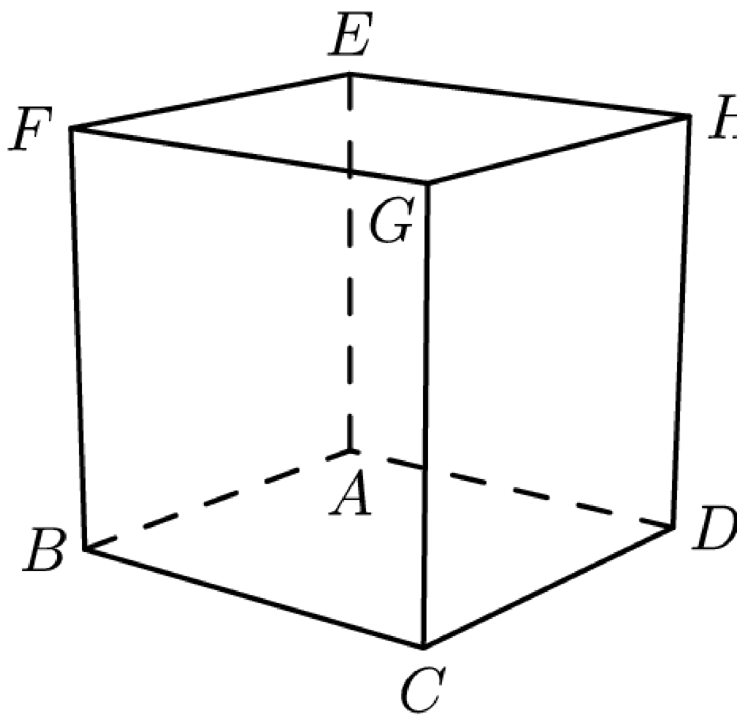
   (c) Finally, to compute the number of permutations containing three instances of $MOOR$, we observe that there is only one possible way of doing so: $MOORMOORMOOR$. Hence, there is only one permutation.

   Using the Principal of Inclusion-Exclusion, the total number of permutations with at least one $MOOR$ is $3780 - 180 + 1 = \boxed{3601}$.

7. Suppose that the 8 corners of a cube can be colored either red, green, or blue. We call a coloring of the cube rotationally symmetric if the cube can be rotated along a single axis parallel to an edge of a cube either 90°, 180°, or 270°, and reach the original coloring. How many rotationally symmetric colorings exist using the 3 colors? Assume that any colorings which are identical after rotation are equivalent.

Answer: 27

Solution:



Since colorings which are identical after rotation are equivalent, we only consider the case where we rotate the cube about the vertical axis (parallel to AE, BF, etc.). Hence, both $ABCD$ and $EFGH$ must have a rotationally symmetric coloring. Moreover, if a cube is rotationally symmetric after rotating by 90° or 270°, it must also be rotationally symmetric after rotating by 180°. Hence, the corners along the diagonals of $ABCD$ or $EFGH$ must be the same color. We proceed using casework on the number of colors:

(a) Case 1: The cube is colored with one color. Then, it can be colored entirely one of the three given colors, for a total of three different colorings.

(b) Case 2: The cube is colored with two colors. Then, one of the following is true:
- $ABCD$ and $EFGH$ are colored two separate colors.
- AC and EG are colored one color, and the others are colored differently.
- AC and FH are colored one color, and the others are colored differently.
- $ABCD$ and EG are colored one color, and the others are colored differently.

In the first three cases, there are three ways to choose the colors, while in the last case there are $3 * 2 = 6$ ways to choose the colors, for a total of $3 * 3 + 6 = 15$ colorings.

(c) Case 3: The cube is colored with three colors. Then, one of the following is true:
- $ABCD$ is colored one color, EG is a second color, and FH is a third color.
- AC and EG are one color, FH is a second color, and BD is a third color.
- AC and FH are one color, EG is a second color, and BD is a third color.

In each case there are three ways to choose the color which is used for the four corners, for a total of $3 * 3 = 9$ colorings.

Hence, there are a total of $3 + 15 + 9 = \boxed{27}$ different rotationally symmetric colorings.

8. Let $x = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + ... + \frac{1}{999999999}$. Compute the number of digits in the first 3000 decimal places of the base 10 representation of x which are greater than or equal to 8.

Answer: 15

Solution: Observe that $\frac{1}{9}$ contributes a 1 in every decimal place; $\frac{1}{99}$ contributes a 1 every two decimal places; $\frac{1}{999}$ contributes a 1 every three decimal places; and so on. Hence, the value of the digit in the nth decimal place indicates how many of the integers $1, ..., 9$ divide n. It suffices to compute the number of integers from 1 to 3000 which divide at least 8 of the first 9 positive integers.

The greatest common multiple of $1, ..., 9$ is $2520 = 2^3 3^2 5^1 7^1$. The greatest common multiple of 8 of the first 9 positive integers can be one of 4 values, if we eliminate 8, 9, 5, or 7 from the set:

$$\begin{aligned} 1260 &= 2^2 3^2 5^1 7^1 \\ 840 &= 2^3 3^1 5^1 7^1 \\ 504 &= 2^3 3^2 5^0 7^1 \\ 360 &= 2^3 3^2 5^1 7^0. \end{aligned}$$

1260 has 2 multiples less than 3000, 840 has 3 multiples less than 3000, 504 has 5 multiples less than 3000, and 360 has 8 multiples less than 3000. Adding these values and subtracting by 3 to account for the fact that 2520 is a common multiple of 1260, 840, 504, and 360, we compute the answer to be $2 + 3 + 5 + 8 - 3 = \boxed{15}$.

9. Two 20-sided dice are rolled. Their outcomes are independent and take uniformly distributed integer values from 1 to 20, inclusive. For each roll, let x be (the sum of the dice) $\times$ (the positive difference of the dice). What is the expected value of x?

Answer: 2793/20 or 139.65

Solution: Consider each of the 400 combinations. In 20 of them, die 1 rolls i. Then x for those rolls are $(i + j)|i - j| = |i^2 - j^2|$, where j is the value of die 2. Since $|i^2 - j^2| = |j^2 - i^2|$, we get

$$\sum_{i,j=1}^{20} |i^2 - j^2| = 2 \sum_{j=1}^{20} \sum_{i=1}^{j-1} (j^2 - i^2).$$

That is, the whole sum is twice the sum restricted to $j > i$, since the terms when $j = i$ are 0. Therefore, the expected value is

$$\frac{1}{400} \cdot 2 \sum_{j=1}^{20} \sum_{i=1}^{j-1} (j^2 - i^2) = \frac{1}{200} \sum_{j=1}^{20} \left(j^2(j-1) - \frac{(j-1)(j)(2j-1)}{6} \right)$$

by the sum of consecutive squares formula. This becomes

$$\frac{1}{200} \sum_{j=1}^{20} \left(\frac{2j^3}{3} - \frac{j^2}{2} - \frac{j}{6} \right).$$

Applying the sum of consecutive integers $\left(\sum_{k=1}^{n} k = \frac{n(n+1)}{2}\right)$, squares $\left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\right)$, and cubes $\left(\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}\right)$ formulas gives:

$$\frac{1}{200}\left(\frac{2}{3}\cdot\frac{20^2\cdot 21^2}{4} - \frac{1}{2}\cdot\frac{20\cdot 21\cdot 41}{6} - \frac{1}{6}\cdot\frac{20\cdot 21}{2}\right) = \frac{27930}{200} = \boxed{\frac{2793}{20}}.$$

10. Compute

$$\sum_{a=1}^{1000}\sum_{b=1}^{1000}\sum_{c=1}^{1000}\left\lfloor\frac{1000}{\operatorname{lcm}(a,b,c)}\right\rfloor \varphi(a)\,\varphi(b)\,\varphi(c)$$

where $\varphi(n) = |\{k : 1 \le k \le n, \gcd(k,n) = 1\}|$ counts the integers coprime to n that are less than or equal to n.

Answer: 250, 500, 250, 000

Solution: We may write

$$\begin{aligned}
\sum_{a=1}^{1000}\sum_{b=1}^{1000}\sum_{c=1}^{1000}\left\lfloor\frac{1000}{\operatorname{lcm}(a,b,c)}\right\rfloor \varphi(a)\,\varphi(b)\,\varphi(c) &= \sum_{a=1}^{1000}\sum_{b=1}^{1000}\sum_{c=1}^{1000}\sum_{k=1}^{\left\lfloor\frac{1000}{\operatorname{lcm}(a,b,c)}\right\rfloor} \varphi(a)\,\varphi(b)\,\varphi(c)\\
&= \sum_{k=1}^{1000}\sum_{a|k}\sum_{b|k}\sum_{c|k}\varphi(a)\,\varphi(b)\,\varphi(c)\\
&= \sum_{k=1}^{1000}\left[\sum_{a|k}\varphi(a)\right]\left[\sum_{b|k}\varphi(b)\right]\left[\sum_{c|k}\varphi(c)\right]\\
&= \sum_{k=1}^{1000}\left[\sum_{d|k}\varphi(d)\right]^3 = \sum_{k=1}^{1000} k^3\\
&= \frac{1000^2\cdot 1001^2}{4} = 250,000\cdot 1002001\\
&= \boxed{250,500,250,000}.
\end{aligned}$$

The second equality is due to reordering the order of summation. We also use

$$\sum_{d|n}\varphi(d) = n.$$

Indeed, $\{1,2,\ldots,n\}$ can be partitioned into the sets $S_d = \{k : 1 \le k \le n, \gcd(k,n) = d\}$ for $d|n$. Noting that $\gcd(k,n) = d \Leftrightarrow \gcd(k/d, n/d) = 1$ we get $|S_d| = \varphi(n/d)$. Thus

$$n = \sum_{d|n}\varphi(n/d) = \sum_{d|n}\varphi(d)$$

as claimed.