1. Kevin plants corn and cotton. Once he harvests the crops, he has 30 pounds of corn and x pounds of cotton. Corn sells for \$5 per pound and cotton sells for \$10 per pound. If Kevin sells all his corn and cotton for a total of \$640, then compute x.

   Answer: 49

   Solution: Kevin sells his corn for $30 \cdot 5$ dollars and his cotton for $x \cdot 10$ dollars. Thus $30 \cdot 5 + x \cdot 10 = 640$, so $x = \frac{640 - 30 \cdot 5}{10} = \frac{490}{10} = \boxed{49}$.

2. $ABCD$ is a square where $AB = \sqrt{2016}$. Let X be a point on AB and Y be a point on CD such that $AX = CY$. Compute the area of trapezoid $AXYD$.

   Answer: 1008

   Solution: Note that trapezoids $AXYD$ and $BXYC$ are congruent, so the area of $AXYD$ is always $\dfrac{2016}{2} = \boxed{1008}$.

3. If the integer n leaves a remainder of 4 when divided by 5, then what is the largest possible remainder when $2n$ is divided by 15?

   Answer: 13

   Solution: Since n leaves a remainder of 4 when divided by 5, it leaves a remainder of 4, 9, or 14 when divided by 15. Thus $2n$ leaves a remainder of $2 \cdot 4 = 8$, $2 \cdot 9 - 15 = 3$, or $2 \cdot 14 - 15 = 13$ when divided by 15, and the maximum remainder is $\boxed{13}$.

4. Let $d(n)$ represent the sum of the digits of the integer n. For example,

   $$d(2016) = 2 + 0 + 1 + 6 = 9.$$

   For how many positive 3-digit integers k is it true that $d(k) > d(k+1)$?

   Answer: 90

   Solution: Note that $d(k) > d(k+1)$ if and only if the units digit of k is 9, because if the units digit of k is not 9, then $d(k) = d(k+1) - 1 < d(k+1)$. Since there are 9 possible hundreds digits and 10 possible tens digits, there are $9 \cdot 10 = \boxed{90}$ possible values of k.

5. Let A, B, and C be three points on circle O such that AC is a diameter of O. Extend line AC to a point D such that DB is tangent to O at B, and suppose that $\angle ADB = 20^\circ$. Compute $\angle ACB$.

   Answer: 55°

   Solution: *Since we did not specify in which direction to extend the line AC, we also accepted 35° as an acceptable answer.*

   Let O be the center of the circle. Through angle chasing, we can determine that

   $$\angle BOD = 90^\circ - \angle ADB = 70^\circ$$
   $$\angle ACB = \frac{180^\circ - \angle BOD}{2} = \boxed{55^\circ}.$$

6. A group of n people, including Caroline and her best friend Catherine, stand in random order around a circle, with each order being equally likely. If the probability that Catherine is adjacent to Caroline is $\frac{1}{9}$, then what is the value of n?

Answer: 19

Solution: Caroline is adjacent to 2 people, and there are $n-1$ people not including Caroline. Therefore, the probability that Catherine is adjacent to Caroline is $\frac{2}{n-1} = \frac{1}{9}$, and solving gives $n = \boxed{19}$.

7. The polynomial $P(x) = x^4 + 4x^3 + 8x^2 + 8x + 4$ is the square of another polynomial $Q(x) = ax^2 + bx + c$ with positive coefficients a, b, and c. Compute $4a + 2b + c$.

Answer: 10

Solution: Note that $Q(2) = 4a+2b+c$ and $P(2) = [Q(2)]^2 = 2^4 + 4\cdot 2^3 + 8\cdot 2^2 + 8\cdot 2 + 4 = 100$. Since a, b, and c are positive, we have $Q(2) > 0$. Taking the positive square root, we get $Q(2) = \sqrt{P(2)} = \sqrt{100} = \boxed{10}$.

8. A physics class has 25 students, two of whom are Alex and Justin. Three students are chosen uniformly at random for a lab demonstration. What is the probability that at least one of Alex and Justin is chosen?

Answer: $\frac{23}{100}$ or 0.23

Solution: We first calculate the probability that neither Alex nor Justin is chosen. The number of ways to choose 3 people in the class is $\binom{25}{3}$. If neither Alex nor Justin can be chosen, there are 23 students that can be chosen and hence $\binom{23}{3}$ ways to choose them. Therefore, the probability that neither Alex nor Justin is chosen is

$$\frac{\binom{23}{3}}{\binom{25}{3}} = \frac{\frac{23\cdot 22\cdot 21}{3!}}{\frac{25\cdot 24\cdot 23}{3!}} = \frac{23\cdot 22\cdot 21}{25\cdot 24\cdot 23} = \frac{22\cdot 21}{25\cdot 24} = \frac{77}{100},$$

so the probability that at least one of them is chosen is

$$1 - \frac{77}{100} = \boxed{\frac{23}{100}}.$$

9. Natasha carries a paintbrush of length 1 and walks around the perimeter of a regular 2016-gon with side length 1, painting all of the area outside the polygon that she can reach. What is the area that Natasha paints?

Answer: $2016 + \pi$

Solution: If the curve is convex, Natasha will only be turning one way, and by the time she returns to the origin she will have turned exactly 2π radians. Therefore, she will paint the area of a circle as she turns corners, because she paints a sector of the unit circle each time until the sum of the sector angles is 2π. On each of the 2016 straight edges, she paints a rectangle of length 1 and width 1, because the 2016-gon has side length 1 and the paintbrush also has length 1. Thus the total area is $2016\cdot 1 + \pi\cdot 1^2 = \boxed{2016+\pi}$.

10. Let S be the set of values which can be written as the sum of five consecutive perfect squares. What is the smallest element of S which is divisible by 17?

Answer: 255

Solution: Each element of S can be written in the form

$$\begin{aligned}(x-2)^2+(x-1)^2+x^2+(x+1)^2+(x+2)^2 &= x^2+2x^2+2+2x^2+8\\ &= 5x^2+10\\ &= 5(x^2+2)\end{aligned}$$

where x is some positive integer with $x \geq 3$. By iterating over values of x, we observe that when $x=7$, $x^2+2=51=3*17$; no smaller value of x exists which satisfies the problem statement. Hence, the solution is $5(x^2+2)=5*51=\boxed{255}$.

11. Consider 6 points, two of which are A and B, such that each pair of points is connected by a line segment and no three points are collinear. Compute the number of distinct paths from A to B such that no point is visited more than once.

Answer: 65

Solution: We consider cases based on the length of the path. Note that there is 1 path of length 1, 4 paths of length 2, $4 \cdot 3 = 12$ paths of length 3, $4 \cdot 3 \cdot 2 = 12$ paths of length 4, $4 \cdot 3 \cdot 2 \cdot 1 = 24$ paths of length 5, and no paths of length 6 or greater. Therefore, there are a total of $1+4+12+24+24=\boxed{65}$ distinct paths.

12. Let $f(x)=3x^2+2x+6$. Two distinct lines l_1 and l_2 exist that are tangent to $f(x)$ and intersect at the origin. Given that l_1 is tangent to $f(x)$ at $(x_1, f(x_1))$ and that l_2 is tangent to $f(x)$ at $(x_2, f(x_2))$, compute x_1x_2.

Answer: −2

Solution: At any point (x_0, y_0) in $f(x)$, the slope of the tangent line is

$$m = 6x_0 + 2$$

Hence, the equation of the tangent line is given by

$$\begin{aligned}(y-y_0) &= m(x-x_0)\\ y-(3x_0^2+2x_0+6) &= (6x_0+2)(x-x_0)\\ y &= (6x_0+2)x-6x_0^2-2x_0+3x_0^2+2x_0+6\\ y &= (6x_0+2)x-3x_0^2+6\end{aligned}$$

For the tangent line to cross the origin, we must have $3x_0^2-6=0$. This equation has two solutions, the product of which is given by $\frac{-6}{3}=\boxed{-2}$.

13. One day, Connie plays a game with a fair 6-sided die. Connie rolls the die until she rolls a 6, at which point the game ends. If she rolls a 6 on her first turn, Connie wins 6 dollars. For each subsequent turn, Connie wins $\frac{1}{6}$ of the amount she would have won the previous turn. What is Connie's expected earnings from the game?

Answer: $\mathbf{\frac{36}{31}}$

Solution: Connie has a $\frac{1}{6}$ chance of winning 6 dollars her first turn. She has a $\frac{5}{6}\frac{1}{6}$ chance of winning 1 dollar her second turn. Next, she has a $\frac{25}{36}\frac{1}{6}$ chance of winning $\frac{1}{6}$ dollars her third turn.

Generalizing, Connie's expected earnings form a geometric series with initial term $\frac{1}{6} * 6 = 1$ and common ratio $\frac{5}{6} * \frac{1}{6} = \frac{5}{36}$. Hence, Connie's expected earnings are

$$\frac{1}{1 - \frac{5}{36}} = \boxed{\frac{36}{31}}$$

14. Calculate the positive root of the polynomial $(x-1)(x-2)(x-3)(x-4) - 63$.

Answer: $\mathbf{\frac{5+\sqrt{37}}{2}}$

Solution: Note that

$$\begin{aligned}(x-1)(x-2)(x-3)(x-4) - 63 &= [(x-2)(x-3)][(x-1)(x-4)] - 63 \\ &= (x^2-5x+6)(x^2-5x+4) - 63 \\ &= [(x^2-5x+5+1)(x^2-5x+5-1)] - 63 \\ &= [(x^2-5x+5)^2 - 1] - 63 \\ &= (x^2-5x+5)^2 - 64 \\ &= (x^2-5x+5+8)(x^2-5x+5-8) \\ &= (x^2-5x+13)(x^2-5x-3).\end{aligned}$$

The polynomial $x^2 - 5x + 13$ has no real roots, while the polynomial $x^2 - 5x - 3$ has roots $x = \frac{5 \pm \sqrt{37}}{2}$. Since $\frac{5-\sqrt{37}}{2} < 0$, the only positive root is $\boxed{\frac{5+\sqrt{37}}{2}}$.

15. Consider an equilateral triangle $\triangle ABC$ with unit side length. Let M be the midpoint of side AB. A ball is released in a straight line from M and bounces off the side BC at a point D. Then, it bounces off the side CA at a point E and lands exactly at B. By the law of reflection, we have $\angle BDM = \angle CDE$ and $\angle CED = \angle AEB$. Calculate $MD + DE + EB$, the distance that the ball travels before reaching B.

Answer: $\mathbf{\frac{\sqrt{13}}{2}}$ **or** $\mathbf{\sqrt{\frac{13}{4}}}$

Solution: The solution can be obtained by reflecting the triangle across edge CB to obtain another equilateral triangle $\triangle A'CB$, and then across $A'C$ to get $\triangle A'B'C$. The length of the line MB' is the minimum distance from M to the vertex B that hits both sides BC and CA'. The length of the line segment BB' is $\frac{\sqrt{3}}{2}$. Thus, the length of $MB' = \sqrt{1 + \frac{3}{4}} = \boxed{\frac{\sqrt{13}}{2}}$

16. Suppose that 12 points are marked evenly around a unit circle. How many pairs of distinct chords are there, with each chord having endpoints among the 12 points, such that the chords are parallel?

Answer: 150

Solution: There are two possible cases:

(a) When all parallel chords are drawn, no unconnected points remain on the unit circle. First, we only consider chords which are vertical. Then, there are $\frac{12}{2} = 6$ chords to select, resulting in $\binom{6}{2} = 15$ distinct chords. Since these chords can be rotated 5 times to obtain other valid pairs, there are $6 * 15 = 90$ possible pairs.

(b) When all parallel chords are drawn, two unconnected points remain on the unit circle. Again, we only consider chords which are vertical. Then, there are $\frac{10}{2} = 5$ chords to select, resulting in $\binom{5}{2} = 10$ distinct chords. Since these chords can be rotated 5 times to obtain other valid pairs, there are $6 * 10 = 60$ possible pairs.

Hence, there are a total of $\boxed{150}$ possible pairs of chords.

17. In the 3-D coordinate plane, two squares of side length 2, $ABCD$ and $EFGH$, are arranged with the sides of $ABCD$ lying parallel to the x- and y-axes and the diagonals of $EFGH$ lying along the y- and z-axes. In addition, the two squares share the same center point, A, B, G, C, D, E are coplanar, and FH is orthogonal to $ABCD$. Compute the volume of the solid $ABGHEFCD$.

Answer: $\frac{8+4\sqrt{2}}{3}$

Solution: The area of the solid can be decomposed into the area of six pyramids: $FABCD$, $FBGC$, $FAED$, $HABCD$, $HBGC$, and $HAED$. Each of these pyramids has height $\sqrt{2}$ and a base of either $2 * 2 = 4$ or $\frac{2*(\sqrt{2}-1)}{2} = \sqrt{2} - 1$. Hence, the total volume of the solid is

$$\begin{aligned} V &= 2 * \frac{1}{3} * \sqrt{2} * 4 + 4 * \frac{1}{3}\sqrt{2} * (\sqrt{2} - 1) \\ &= \frac{8}{3}\sqrt{2} + \frac{4}{3}(2 - \sqrt{2}) \\ &= \frac{8}{3}\sqrt{2} + \frac{8}{3} - \frac{4}{3}\sqrt{2} \\ &= \boxed{\frac{8+4\sqrt{2}}{3}}. \end{aligned}$$

18. Let F be the Fibonacci sequence of numbers, where $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. The sum $\sum_{i=1}^{2016} F_{3i}$ can be written in the form $\frac{1}{a}(F_b - c)$, where a, b, c are positive integers, $a > 1$, and b is minimal. Compute $a + b + c$.

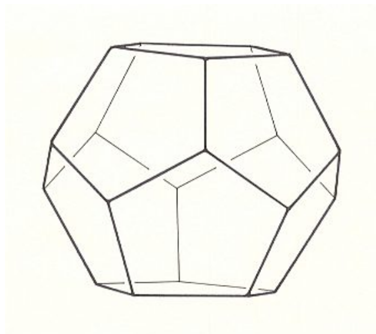
Answer: 6053

Solution: Note that the sum can be expressed as

$$\begin{aligned} \sum_{i=1}^{2016} F_{3i} &= \frac{1}{2}\sum_{i=1}^{2016} 2F_{3i} \\ &= \frac{1}{2}\sum_{i=1}^{2016}(F_{3i} + F_{3i-1} + F_{3i-2}) \\ &= \frac{1}{2}\left[(\sum_{i=1}^{6048} F_i) + F_2 - F_2\right] \\ &= \frac{1}{2}(F_{6050} - 1) \end{aligned}$$

where in the 3rd step, the sum $(\sum_{i=1}^{6048} F_i) + F_2$ telescopes to the single term F_{6050}. Hence, $a + b + c = 2 + 6050 + 1 = \boxed{6053}$.

19. A regular dodecahedron is a figure with 12 identical pentagons for each of its faces. Let x be the number of ways to color the faces of the dodecahedron with 12 different colors, where two colorings are identical if one can be rotated to obtain the other. Compute $\frac{x}{12!}$.



Answer: $\mathbf{\frac{1}{60}}$

Solution: First, we fix one color for the top of the figure. Then, there are 11 possible choices for the color of the face directly opposite the top. Since there are 10 remaining faces, we can color them a total of 10! possible ways. However, since the figure can be rotated about the vertical axis five times to obtain 5 different orientations, we must divide by 5 to prevent overcounting. Hence, the desired answer is

$$\frac{x}{12!} = \frac{11 * 10!/5}{12!} = \frac{12 * 11 * 10!}{12 * 5 * 12!} = \boxed{\frac{1}{60}}$$

20. Compute

$$\int_0^3 \sqrt{\sqrt{x+1}-1}\, dx.$$

Answer: $\mathbf{\frac{32}{15}}$

Solution: We make the substitution $z = \sqrt{\sqrt{x+1}-1}$, so that

$$x = (z^2+1)^2 - 1,\ dx = 4z(z^2+1)\, dz,$$

In addition, the endpoints map to 0 and 1. Then the integral can be written as

$$\int_0^1 z \cdot (4z(z^2+1)\, dz) = \int_0^1 (4z^4 + 4z^2)\, dz = \frac{4}{5} + \frac{4}{3} = \boxed{\frac{32}{15}}.$$

21. Let ω be a root of the polynomial $\omega^{2016} + \omega^{2015} + \cdots + \omega + 1$. Compute the sum

$$\sum_{i=0}^{2016} (1+\omega^i)^{2017}.$$

Answer: 4034

Solution: Using the Binomial Theorem, we expand $(1+\omega^i)^{2017}$:

$$\sum_{i=0}^{2016}(1+\omega^i)^{2017} = \sum_{i=0}^{2016}\sum_{j=0}^{2017}\binom{2017}{j}\omega^{ij}.$$

We can then swap the indices of the sums and move out the $\binom{2017}{j}$ to get

$$\sum_{i=0}^{2016}\sum_{j=0}^{2017}\binom{2017}{j}\omega^{ij} = \sum_{j=0}^{2017}\binom{2017}{j}\sum_{i=0}^{2016}\omega^{ij}.$$

Now consider the sum $\sum_{i=0}^{2016}\omega^{ij}$ for j fixed. Notice that $w^{2017} = 1$ and 2017 is a prime. Therefore, if $\omega^{aj} = \omega^{bj}$ for some $0 \le a, b \le i$, then $\omega^{|a-b|j} = 1$, hence either $a = b$ or $j \mid 2017$.

If $j \mid 2017$, then $\omega^{ij} = (\omega^j)^i = 1^i = 1$ for all i, so $\sum_{i=0}^{2016}\omega^{ij} = 2017$.

Otherwise, if $j \nmid 2017$, then each of the ω^{ij} are distinct, so $\sum_{i=0}^{2016}\omega^{ij}$ is just a permutation of the sum $\sum_{i=0}^{2016}\omega^i = 0$ from our condition. Therefore, $\sum_{i=0}^{2016}\omega^{ij} = 0$.

Therefore, we have

$$\sum_{j=0}^{2017}\binom{2017}{j}\sum_{i=0}^{2016}\omega^{ij} = \left(\binom{2017}{0}+\binom{2017}{2017}\right)\cdot 2017 = \boxed{4034}.$$

22. A lattice point is a pair (x, y) where x and y are integers. How many lattice points are on the graph of $y = \sqrt{x^2+60^4}+2016$?

Answer: 175

Solution: Let $z = y - 2016 = \sqrt{x^2+60^4}$. Note that the graphs of y and z have the same number of lattice points, since z is just y translated down 2016 units. Then $z^2 = x^2 + 60^4$, or

$$(z+x)(z-x) = 60^4 = 2^8 3^4 5^4.$$

Because z equals a square root, we have $z \ge 0$. Since x is an integer, $z+x$ and $z-x$ must have the same parity. However, they cannot be both odd, because their product is 60^4, an even number. Thus $z+x$ and $z-x$ are both even. Now, if $z+x$ is negative, then $z-x = \frac{60^4}{z+x}$ is negative as well, so $(z+x)+(z-x) = 2z < 0$, which contradicts $z \ge 0$. Thus $z+x > 0$, which implies $z - x > 0$.

Now, suppose $z + x = 2^a 3^b 5^c$, where a, b, and c are nonnegative integers. If $a = 0$, then $z+x$ is odd, which is not allowed, and if $a = 8$, then $z-x$ is odd, which is also not allowed. Thus $1 \le a \le 7$. Since $z = \frac{(z+x)+(z-x)}{2}$ and $x = \frac{(z+x)-(z-x)}{2}$, only the parities of $z+x$ and $z-x$ restrict solutions. Thus b and c are unrestricted, so $0 \le b \le 4$ and $0 \le c \le 4$. Therefore, there are $7 \cdot 5 \cdot 5 = 175$ possible ordered triples (a, b, c).

Each (a, b, c) corresponds to a unique value of $z + x$, each of which uniquely determines $z - x$. Then, z and x are uniquely determined from $z = \frac{(z+x)+(z-x)}{2}$ and $x = \frac{(z+x)-(z-x)}{2}$, and finally, y is uniquely determined from $y = z+2016$. We conclude that there is a one-to-one correspondence between solutions (a, b, c) and lattice points (x, y). Hence, there are $\boxed{175}$ lattice points.

23. Compute
$$\sum_{\substack{p,q+1\geq 1\\ \gcd(p,q)=1}} \frac{1}{3^{p+q}-1}.$$

Answer: $\mathbf{\frac{3}{4}}$ **or 0.75**

Solution: We begin by computing
$$\sum_{\substack{p,q+1\geq 1\\ \gcd(p,q)=1}} \frac{1}{3^{p+q}-1} = \sum_{d=1}^{\infty} \frac{1}{3^d-1} \sum_{\substack{p,q+1\geq 1\\ \gcd(p,q)=1\\ p+q=d}} 1 = \sum_{d=1}^{\infty} \frac{\varphi(d)}{3^d-1}$$
where $\varphi(d) = |\{k : 1 \leq k \leq d, \gcd(k,d) = 1\}|$. This is true because
$$p, q+1 \geq 1, \gcd(p,q) = 1, p+q = d \Leftrightarrow 1 \leq p \leq d, \gcd(p,d) = 1, q = d-p,$$
so
$$\sum_{\substack{p,q+1\geq 0\\ \gcd(p,q)=1\\ p+q=d}} 1 = \sum_{\substack{1\leq p\leq d\\ \gcd(p,d)=1}} 1 = \varphi(d)$$
simply by the definition of φ. Next, we observe that
$$\sum_{d=1}^{\infty} \frac{\varphi(d)}{3^d-1} = \sum_{d=1}^{\infty} \varphi(d) \frac{3^{-d}}{1-3^{-d}} = \sum_{d=1}^{\infty} \varphi(d) \sum_{k=1}^{\infty} 3^{-dk} =$$
$$= \sum_{d=1}^{\infty} \sum_{k=1}^{\infty} 3^{-dk}\varphi(d) = \sum_{d=1}^{\infty} \sum_{\substack{n\geq 1\\ n=dk}} 3^{-n}\varphi(d) = \sum_{n=1}^{\infty} \sum_{d|n} 3^{-n}\varphi(d),$$
where the last equality is the result of interchanging the order of summation. Next, we show
$$\sum_{d|n} \varphi(d) = n.$$
Indeed, $\{1, 2, \ldots, n\}$ can be partitioned into the sets $S_d = \{k : 1 \leq k \leq n, \gcd(k,n) = d\}$ for $d|n$. Noting that $\gcd(k,n) = d \Leftrightarrow \gcd(k/d, n/d) = 1$ we get $|S_d| = \varphi(n/d)$. Thus
$$n = \sum_{d|n} \varphi(n/d) = \sum_{d|n} \varphi(d)$$
as claimed. Thus, we compute
$$\sum_{n=1}^{\infty} \sum_{d|n} 3^{-n}\varphi(d) = \sum_{n=1}^{\infty} 3^{-n} \sum_{d|n} \varphi(d) = \sum_{n=1}^{\infty} n3^{-n} = \frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \boxed{\frac{3}{4}}.$$
Here we use the well known formula that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$, which we may alternatively derive by computing
$$\sum_{n=1}^{\infty} nx^n = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} x^k = \sum_{n=1}^{\infty} \frac{x^n}{1-x} = \frac{x}{(1-x)^2}.$$

24. Let ABC be a right triangle with $\angle C = 90°$. The angle bisector of $\angle ACB$ intersects AB at D. In addition, the angle bisector of $\angle ADC$ intersects AC at E, and the angle bisector of $\angle BDC$ intersects BC at F. Suppose that $EC = 6$ and $FC = 8$, and let the intersection of EF and CD be G. Compute GD.

Answer: $\frac{25\sqrt{2}}{7}$ **or** $\frac{50}{7\sqrt{2}}$

Solution 1: First, we make some general observations. Note that $\angle ACD = \angle BCD = \frac{\angle ACB}{2} = 45$. In addition, since ED bisects $\angle ADC$ and FD bisects $\angle BDC$, we must have $\angle EDF = \frac{\angle ADC + \angle BDC}{2} = 90$. Finally, since ECF is also a right triangle, we have $EF = \sqrt{6^2 + 8^2} = 10$, by the Pythagorean Theorem.

Next, we compute $x = CD$. Let $c = ED$ and $d = FD$. Using the law of cosines, we can write the following equations

$$c^2 = 6^2 + x^2 - 2 * 6 * x \cos 45$$
$$d^2 = 8^2 + x^2 - 2 * 8 * x \cos 45$$

Since $\angle EDF = 90$, we have $EF^2 = c^2 + d^2$ by the Pythagorean Theorem. Hence, we may write

$$\begin{aligned} EF^2 &= 6^2 + x^2 - 2 * 6 * x \cos 45 + 8^2 + x^2 - 2 * 8 * x \cos 45 \\ 6^2 + 8^2 &= 6^2 + 8^2 + 2x^2 - 6x\sqrt{2} - 8x\sqrt{2} \\ 2x^2 &= (6+8)x\sqrt{2} \\ x &= 7\sqrt{2} \end{aligned}$$

We can then solve for c and d:

$$\begin{aligned} c^2 &= 6^2 + (7\sqrt{2})^2 - \sqrt{2} * 6 * 7\sqrt{2} \\ c &= \sqrt{50} = 5\sqrt{2} \\ d^2 &= 8^2 + (7\sqrt{2})^2 - \sqrt{2} * 8 * 7\sqrt{2} \\ d &= \sqrt{50} = 5\sqrt{2} \end{aligned}$$

Since $c = ED = d = FD$, $\angle DEF = \angle DFE = 45$. Next, since $\angle DCF = \angle GFD = 45$ and $\angle CDF = \angle GDF$, we have $\triangle CDF \sim \triangle FDG$ by AA similarity. We may then write the ratio $\frac{DG}{DF} = \frac{DF}{DC}$. Solving for DG, the desired quantity is

$$DG = \frac{DF^2}{DC} = \frac{(5\sqrt{2})^2}{7\sqrt{2}} = \boxed{\frac{25\sqrt{2}}{7}}$$

Solution 2: Observe that $CEDF$ is a cyclic quadrilateral, since $\angle EDF + \angle ECF = 90 + 90 = 180$. Moreover, since $\angle DCF$ and $\angle DEF$ subtend the same arc, we have $\angle DEF = \angle DCF = 45$. Similarly, $\angle ECD$ and $\angle EFD$ subtend the same arc, and $\angle EFD = \angle ECD = 45$. We conclude that EFD is a 45-45-90 triangle, and since $EF = 10$, $ED = FD = 5\sqrt{2}$.

Since $\angle DCF = \angle GFD = 45$ and $\angle CDF = \angle GDF$, we have $\triangle CDF \sim \triangle FDG$ by AA similarity. Next, since CG bisects $\angle ECF$, we have that $\frac{GF}{EG} = \frac{FC}{CE}$, and equivalently $\frac{GF}{10-GF} = \frac{4}{3}$.

Solving for GF, we have that $GF = \frac{40}{7}$. Finally, from the similarity of the triangles noted above, we obtain the ratio $\frac{FD}{FC} = \frac{GD}{GF}$. Solving for GD, the desired quantity is

$$GD = \frac{FD * GF}{FC} = \frac{5\sqrt{2} * \frac{40}{7}}{8} = \boxed{\frac{25\sqrt{2}}{7}}$$

25. Compute the number of digits in the decimal representation of

$$\prod_{n=1}^{2016} n^n.$$

Your score will be given by $\lfloor 25 \min\{(\frac{A}{C})^2, (\frac{C}{A})^2\}\rfloor$, where A is your answer and C is the actual answer.

Answer: 6277209

Solution: Note that the number of digits is at most 1 away from

$$\log \prod_{n=1}^{2016} n^n = \sum_{n=1}^{2016} \log n^n = \sum_{n=1}^{2016} n \log n.$$

A first approximation might be

$$\sum_{n=1}^{2016} n \log n \approx \sum_{n=1}^{2016} n = \frac{2016 \cdot 2017}{2} \approx \frac{2000 \cdot 2000}{2} = 2000000,$$

which would earn 2 points.

However, we can do much better by noting that $\log n$ is a concave down function, so for $1 \leq n \leq 2016$, most values of $\log n$ are closer to $\log 2016$ than $\log 1$. Therefore, we might try

$$\sum_{n=1}^{2016} n \log n \approx \sum_{n=1}^{2016} n \log 2016 \approx \sum_{n=1}^{2016} n \log 1000 = 3 \sum_{n=1}^{2016} n \approx 6000000,$$

which would earn 22 points.

26. 1000 unit cubes are stacked to form a cube with side length 10. A sphere of radius 10 is centered at the corner of the large cube. How many small cubes are cut by the sphere? In other words, how many small cubes have interior points which lie on the surface of the sphere? Your score will be given by $\lfloor 25 \min\{(\frac{A}{C})^2, (\frac{C}{A})^2\}\rfloor$, where A is your answer and C is the actual answer.

Answer: 229

Solution: We look to the 2-dimensional plane for inspiration. Consider a circle of radius r centered at the origin; we would like to count the number of unit squares intersected by the circle counting from the positive y-intercept to the positive x-intercept. Assuming that the circle contains no lattice points other than at $(0, \pm r)$ and $(\pm r, 0)$, the circle will pass through $2\lceil r \rceil - 1$ unit squares, as the circle must pass through $\lceil r \rceil - 1$ unit squares down and $\lceil r \rceil - 1$ unit squares right to the positive x-intercept. For every lattice point on the circle which is not a x or y-intercept, we decrease the number of unit squares by 1.

Next, consider the 3-dimensional case. Let the center of the sphere lay at the origin, assume WLOG that the unit cubes lie in the positive octant of the 3D space, and consider the section of the sphere lying between $z = a$ and $z = a+1$, where a is a positive integer. The planes $z = a$ and $z = a+1$ intersect the sphere at the circles $x^2 + y^2 = 100 - a^2$ and $x^2 + y^2 = 100 - (a+1)^2$. These cross sections intersect $2\sqrt{100-a^2} - 1$ unit cubes and $2\sqrt{100-(a-1)^2} - 1$ unit cubes, respectively, between the planes $z = a$ and $z = a+1$. If the radii are similar in value, then the cross sections will intersect the same unit cubes; if the radii are significantly different in value, then the cross sections will intersect different sets of unit cubes, and the sphere will intersect not only those sets of unit cubes but also the unit cubes lying between the unit cubes which intersect the cross sections. Finally, to sum up the number of intersected unit cubes, we take length 1 sections of the sphere, compute the approximate radii of the sphere cross sections, and approximate the total number of unit cubes.

The radii of the sphere cross sections are as follows, where we round up for simplicity:

$$\begin{array}{ll}
z = 0: & r = \sqrt{100-0} \approx 10 \\
z = 1: & r = \sqrt{100-1} \approx 10 \\
z = 2: & r = \sqrt{100-4} \approx 10 \\
z = 3: & r = \sqrt{100-9} \approx 10 \\
z = 4: & r = \sqrt{100-16} \approx 10 \\
z = 5: & r = \sqrt{100-25} \approx 9 \\
z = 6: & r = \sqrt{100-36} = 8 \\
z = 7: & r = \sqrt{100-49} \approx 8 \\
z = 8: & r = \sqrt{100-64} = 6 \\
z = 9: & r = \sqrt{100-81} \approx 5 \\
z = 10: & r = \sqrt{100-100} = 0
\end{array}$$

Then, for each section lying between $z = a$ and $z = a+1$ for all a, we add $2r - 1$ if the radii of the two cross sections is equal and otherwise add $2r' - 1$ for integers $r_1 \le r' \le r_2$ if the radii r_1, r_2 of the two cross sections is different. The approximate total number of intersected unit cubes is thus

$$\begin{aligned}
&19 + 19 + 19 + 19 && \text{(sections between } z = 0,\ z = 4) \\
&+ (19 + 17) + (17 + 15) && \text{(sections between } z = 4,\ z = 6) \\
&+ 15 + (15 + 13 + 11) && \text{(sections between } z = 6,\ z = 8) \\
&+ (11 + 9) + (9 + 7 + 5 + 3 + 1) && \text{(sections between } z = 8,\ z = 10) \\
&= 243
\end{aligned}$$

We can obtain a slightly better approximation by taking into consideration lattice points and other adjustments. However, the current guess gives a good score of 22.

27. You play a game where you roll a fair 6-sided die repeatedly until the cumulative sum of your rolls is exactly a perfect cube, at which point the game ends. What is the expected number of rolls before the game ends? Your score will be given by $\lfloor 25 \min\{(\frac{A}{C})^2, (\frac{C}{A})^2\} \rfloor$, where A is your answer and C is the actual answer.

Answer: 63.5773317

Solution: One way to get a reasonably close estimate is to first compute the expected value of the cumulative sum. We say we *pass* a perfect cube n^3 if the cumulative sum exceeds n^3 without reaching exactly n^3, or equivalently, if the game does not end at n^3.

Note that for some $n \geq 1$, the probability of passing n^3 given that we have already passed $(n-1)^3$ is approximately $\frac{5}{6}$, because there is approximately a $\frac{1}{6}$ chance that we land on n^3 exactly. Thus if E is the expected number of perfect cubes that we pass, then $E = \frac{5}{6}(E+1) + \frac{1}{6}$, or $E = 6$. So the expected cumulative sum is approximately $6^3 = 216$. Since the expected value of a dice roll is 3.5, the expected number of rolls before the game ends is $\frac{216}{3.5} \approx 62$, which is quite close to the actual value and gives a good score of 23.