Time limit: 90 minutes.
Maximum score: 200 points.
Instructions: For this test, you work in teams of eight to solve a multi-part, proof-oriented question.

Problems that use the words "compute", "list", or "draw" only call for an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof. Answers should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Place a team ID sticker on every submitted page. If you do not have your stickers, you should write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your packet. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them.

The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.

No calculators.

Introduction

This power round is about graph theory, and in particular, about graph colorings. In this case we refer to graphs as those with vertices and edges, and not those which are plots of functions. Usually, when we discuss colorings of a graph, we are concerned with assigning colors to the vertices or edges of the graph such that certain properties are satisfied. For example, we might be interested in colorings of graphs where no two adjacent vertices have the same color. This problem has numerous applications in scheduling and the development of compilers, and also has interesting applications to networking and parallelization of tasks. For example, if we need to complete a certain number of tasks which compete for resources, we can let vertices be tasks and let edges represent a conflict between two tasks for a certain resource. A coloring of this graph then represents a partition of the tasks into sets that can be accomplished simultaneously, with the number of colors used representing the number of instances we must assign tasks to resources.

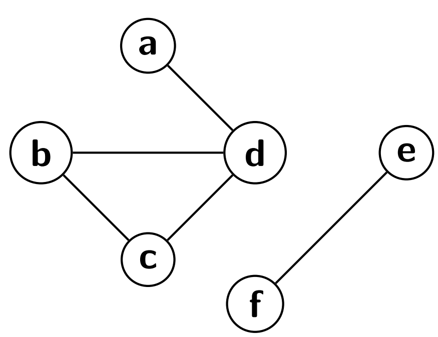
Graph colorings are a lot of fun and easily visualizable. We hope you enjoy working with these problems and that you leave with more experience and interest in the theory and applications of graph colorings.

Graph Basics

A caveat: In this section, we define a graph and many of its important features and properties. Although there are many terms and definitions in this section, they are meant to facilitate writing about graphs, and do not have much useful content in and of themselves. **We advise that you do not spend too much time reading and memorizing this first section, and instead treat it as a glossary for you to reference when in need of helpful language.**

Definition. A **graph** G is a pair (V, E) where V is a set of **vertices** and E is a set of **edges**. Each edge is a set $\{x, y\}$ where x and y are distinct vertices in V. We will usually denote such an edge as xy or yx . If such an edge exists, x and y are then said to be **adjacent** or **neighbors**. For a graph G, we refer to its set of vertices as $V(G)$ and its set of edges as $E(G)$. Additionally, we will define the number of vertices of G as its **order**. We disallow self-loops (edges connecting a vertex to itself) and multiple edges connecting the same pair of vertices.

Example.



A simple graph with 6 nodes and 5 edges.

Definition. A **complete** graph is one where every vertex is adjacent to every other vertex. We will denote the complete graph with n vertices as K^n.

Definition. The **degree** of a vertex is the number of edges incident on (or attached to) that vertex. For a vertex v in a graph G, we denote its degree with $d(v)$. The **maximum degree** (denoted by $\Delta(G)$) of a graph G is the largest degree of any vertex in G, and the **minimum degree** (denoted by $\delta(G)$) is the smallest degree of any vertex in G.

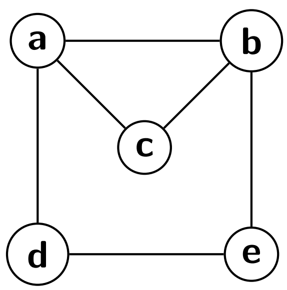
Example. The complete graph K_5 has $d(v) = \Delta(G) = \delta(G) = 4$ for every vertex v. In general, every vertex v in $V(K^n)$ has $d(v) = \Delta(G) = \delta(G) = n - 1$.

Definition. A **subgraph** G' of a graph $G = (V, E)$ is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. Informally, it is a graph obtained by taking G and selecting a set of edges and vertices such that the selection is still a valid graph. Notationally, we reuse the subset notation and say that $G' \subseteq G$ indicating that G' is a subgraph of G.

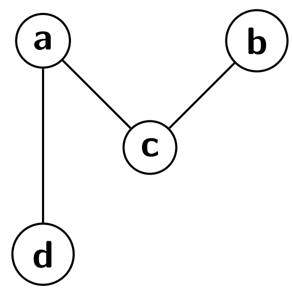
An **induced subgraph** G' is formed by selecting a subset V' of the vertices of G and then taking E' to be the set of edges between vertices in V' that are also in E.

For notational convenience, we will often refer to the graph G with vertex v removed (along with its associated edges) as $G - v$.

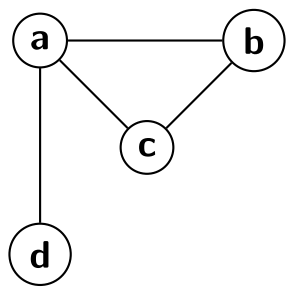
Example.



A graph G.



A subgraph of G.



The subgraph of G induced by nodes a, b, c, and d.

Definition. A **path** in a graph G is a route from one vertex to another, traveling only across edges of the graph. More precisely, it is the subgraph of G described by such a route. We will usually write a path from vertices v_0 to v_k as $P = v_0v_1v_2...v_k$. The length of a path is the number of edges in the path. A **cycle** with vertices $v_0...v_{k-1}$ is a path $v_0...v_{k-1}$ of length greater than 1, with the additional edge $v_{k-1}v_0$. The length of a cycle is the number of vertices (or edges) in the cycle. Note that paths and cycles do not have directions; they are simply graphs with vertices and edges.

Definition. A graph is **connected** if there exists a path from every vertex to every other vertex. A **connected component** or simply **component** of a graph G is an induced subgraph of G that is itself connected and to which no other vertices of G are connected, except the ones in the subgraph.

A note: for a few questions, we will ask you to draw a graph. Additionally, you may find it useful to reference graphs or parts of graphs when writing your proofs. When doing so, please draw the vertices and edges cleanly, and label specific vertices or edges if they are used in your proof. **Be advised that we will not tolerate messy graphs.**

1. (a) i. **[2]** Draw a connected graph G with 8 vertices such that $\Delta(G) = \delta(G) = 3$. (A graph with $\Delta(G) = \delta(G) = 3$ is called a *cubic* graph).

   ii. **[2]** Draw a connected graph G with more than 5 vertices such that $|V| = |E| + 1$. ($|S|$ denotes the number of elements in the set S.) This graph is an example of a special type of graph known as a **tree**.

   iii. **[2]** Draw a connected graph G with 4 vertices and 4 edges that is **bipartite** - that is, we can partition the vertices of G into two classes such that every edge of G has one endpoint in one class and one endpoint in the other.

   (b) **[4]** Prove that for any graph, there must be an even number of vertices of odd degree.

   (c) **[5]** Prove that a connected graph G is bipartite if and only if it contains no cycles of odd length.

   (d) **[6]** Prove that a graph G with $\delta(G) \geq 2$ has a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$. (Hint: Consider the longest path in G.)

   (e) Here, we explore the number of cycles in complete graphs.

   i. **[3]** Compute the number of cycles in K^4. Remember that a cycle is itself defined as a subgraph of G, and two graphs are equal if their edge sets and node sets are equal. You do not need to justify your answer.

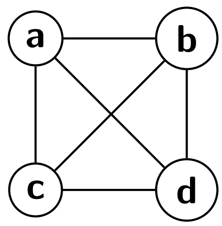
   ii. **[7]** Derive a general formula for the number of cycles in K^n. For full credit, the formula should not be recursive (i.e. should not contain the number of cycles of K^m for $m \leq n$ as a term). You do not need to justify your answer.

Planar Graphs and the 5-Color Theorem

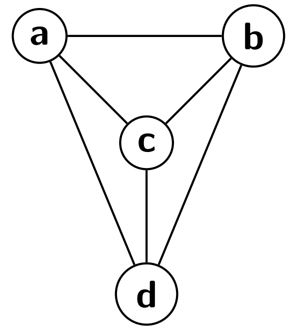
Planar graphs are a particularly interesting category of graphs that have some special coloring properties. One of the most famous theorems about planar graphs is the 4-color theorem, which states that any planar graph needs only 4 or fewer colors to be colored in such a way that adjacent vertices are different colors. It was finally proven in 1977 with the assistance of a computer - the problem was first reduced to a large but finite number of base cases, which the computer then verified. We begin with a few definitions.

Definition. A graph G is **planar** if it can be represented on a piece of paper such that no edges cross each other (or intersect with vertices other than their respective endpoints). See the example below.

Example.



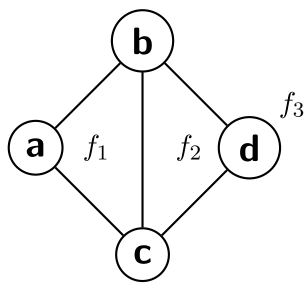
A planar graph drawn (incorrectly) with overlapping edges.



The same planar graph drawn without overlapping edges.

Definition. A **face** of a planar graph is a region in in the plane bounded by the edges and vertices of the graph (so long as the graph is drawn correctly as a planar graph with no intersecting edges). Note that the region "outside" of the graph is also a face. In the following picture, the faces are labeled f_1, f_2, and f_3 where the third face is the unbounded region external to the graph.

Example.



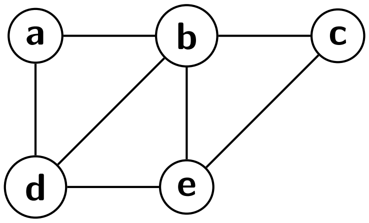
A simple graph.

Theorem (Euler's Formula)**.** *If G is a connected planar graph with f faces, e edges, and v vertices, then $f - e + v = 2$.*

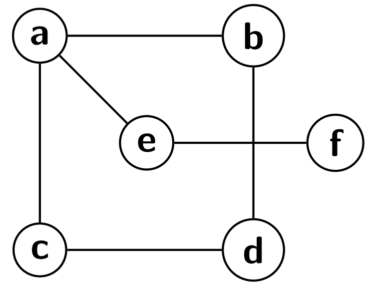
Definition. A **proper coloring** (or simply a **coloring**) of a graph G is a mapping from the vertices of G to colors $\{1, 2, ..., k\}$ such that adjacent vertices are different colors.

2. (a) i. **[2]** Confirm that Euler's Formula holds for graph (i) below, or explain why it doesn't. Show your work.
   ii. **[2]** Confirm that Euler's Formula holds for graph (ii) below, or explain why it doesn't. Show your work.
   iii. **[2]** Confirm that Euler's Formula holds for graph (iii) below, or explain why it doesn't. Show your work.

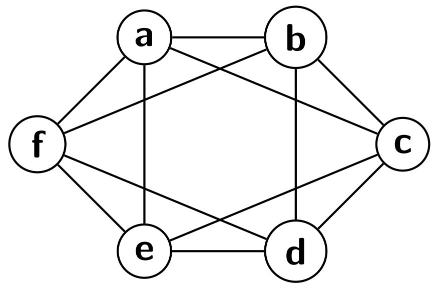
   (b) **[6]** Prove Euler's Formula for planar graphs. (Hint: Use induction on the number of edges. What might happen if you remove an edge from the graph? How will f, e and v be affected?)



Problem 2.a.i.



Problem 2.a.ii.



Problem 2.a.iii.

(c) The **5-color theorem** states that every planar graph has a coloring using a maximum of five colors. This is a relaxation of the 4-color theorem that can be proven with less casework. You **may not** use the 4-color theorem to prove any parts in this section.

i. **[5]** Use Euler's Formula to prove the following: A planar graph with $v \geq 3$ vertices has at most $3v - 6$ edges.

ii. **[5]** Use this to prove that a planar graph must have a vertex of degree ≤ 5.

iii. **[10]** Consider a counterexample to the theorem - a vertex with exactly five neighbors all of which are different colors $\{1, ..., 5\}$. If the vertices are colored from 1 to 5 clockwise, show that vertices colored 2 and 4 cannot be connected by a path containing vertices colored only 2 and 4. Use this fact to complete the proof. (Hint: Start by using induction on the number of vertices.)

The Chromatic Number and Brook's Theorem

Now that we have some basic facts about graphs, it's time to examine one of the central concepts in graph coloring theory: the chromatic number.

Definition. The **chromatic number** of a graph G (denoted $\chi(G)$) is the minimum number of different colors required to color every vertex of G, such that each vertex is colored with exactly one color and adjacent vertices are different colors. We will sometimes refer to a graph G with $\chi(G) = n$ as an n**-colorable** graph.

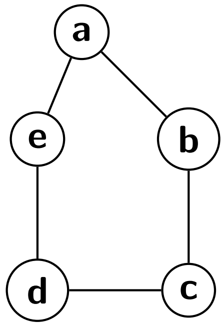
Example. For a complete graph K^n, $\chi(K^n) = n$ because every vertex is adjacent to every other vertex.

Often, the chromatic number is difficult to compute for an arbitrary graph. For example, it may depend on global properties (i.e. large-scale structures) of the graph that may be difficult to capture mathematically. However, there are some very useful bounds on the chromatic number of graphs that we can find using *local* properties of the graph (such as the degree of vertices). We examine some of the more common properties here.

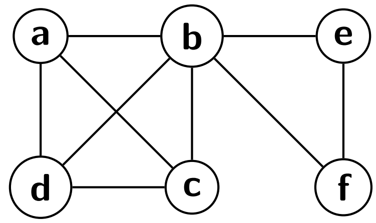
3. (a) i. **[2]** Find a 3-coloring of graph (i) below. Explain why its chromatic number is 3.

ii. **[2]** Find a 4-coloring of graph (ii) below. Explain briefly why its chromatic number is 4.

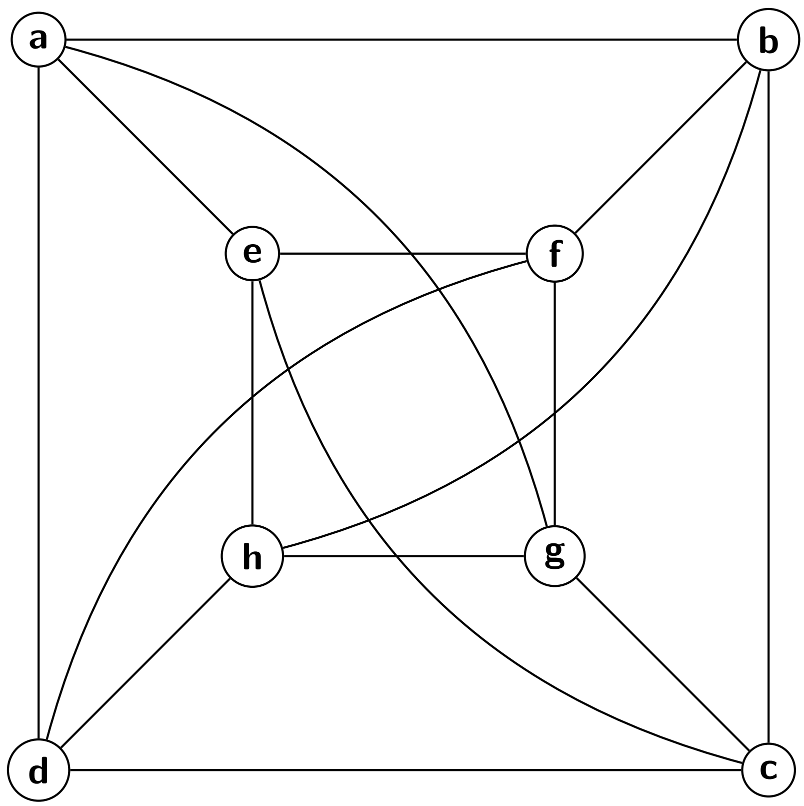
iii. **[2]** Find a 2-coloring of graph (iii) below. Explain briefly why its chromatic number is 2.



Problem 3.a.i.



Problem 3.a.ii.



Problem 3.a.iii.

(b) Sometimes, we can compute the chromatic number directly.

i. **[2]** Find the chromatic number of a connected bipartite graph. Justify your answer.

ii. **[3]** Find the chromatic number of an order $n+1$ **wheel graph** - a cycle of n vertices with an additional "central" vertex adjacent to all of the n vertices in the cycle, totaling $n+1$ vertices and $2n$ edges. Justify your answer.

(c) **[6]** Prove that any graph G with m edges satisfies $\chi(G) \leq 1/2 + \sqrt{2m + 1/4}$

(d) *Brooks' Theorem.* Prove that for any connected graph that is not a complete graph or an odd-length cycle, $\chi(G) \leq \Delta(G)$. This problem is cumulative - if you cannot prove a particular part, you may still use its results in later parts. We recommend drawing sketches and generally trying to visualize the concepts embodied in each section of this proof.

i. **[2]** Show why the theorem does not hold for complete graphs and odd-length cycles.

ii. **[2]** Show why the theorem holds for paths and even-length cycles (the only graphs with $\Delta(G) \leq 2$). This lets us assume that $\Delta(G) \geq 3$ for the following steps.

iii. **[5]** We now apply induction on the number of vertices. Assume all graphs H (except complete graphs and odd-length cycles) of order less than G satisfy $\chi(H) \leq \Delta(H)$. Consider a counterexample G such that $\chi(G) > \Delta(G)$. Remove a vertex v to create $H = G - v$. Show that v must have had exactly $\Delta(G)$ neighbors, all different colors. For future reference, we will call these vertices $v_1, v_2, ..., v_{\Delta(G)}$, colored 1, 2, ..., $\Delta(G)$ respectively.

iv. **[4]** Let $H_{i,j}$ be the induced subgraph of $G - v$ that contains exactly the vertices colored i and j, where $i \neq j$. Show that the vertices colored i and j that are adjacent to v must be in the same component $C_{i,j}$ of $H_{i,j}$.

v. **[6]** Note that v_i must have $\Delta(G) - 1$ neighbors in $G - v$, colored with $\Delta(G) - 1$ different colors, otherwise we could recolor v_i, leading to a contradiction with part iii. Now show that $C_{i,j}$ must be a path of vertices of alternating colors i and j from v_i to v_j. (Hint: Assume for the purpose of contradiction that there is a vertex u colored j on $C_{i,j}$ with three neighbors colored i (or colored i with three neighbors colored j), and show that it is then possible to recolor u).

vi. **[6]** Show that for distinct i, j, and k, $C_{i,j}$ and $C_{i,k}$ can only meet at v_i.

vii. **[10]** Show why the neighbors v_i of v cannot all be pairwise adjacent. Now complete the proof by finding a recoloring of $G - v$ that results in a contradiction of the fact proved in the previous part. (Hint: Consider a neighbor u of v_i that is not a neighbor of v. Can you recolor $C_{i,k}$ for some k in such a way that forces u to be a member of two different paths between neighbors of v?)

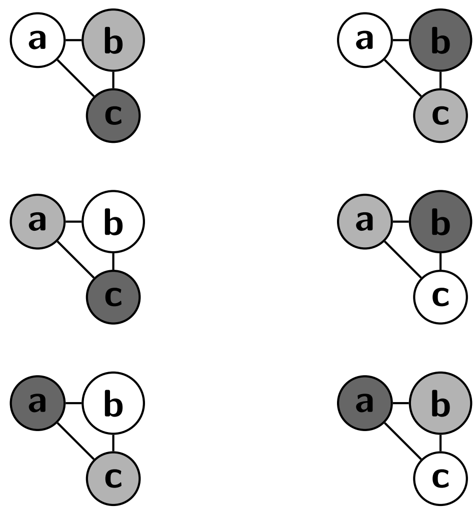
The Chromatic Polynomial

Now we will discuss a generalization of the chromatic number of a graph known as the chromatic polynomial.

Definition. The **chromatic polynomial** $P_G(k)$ of a graph G is a polynomial such that for each k, $P_G(k)$ gives the number of colorings of G that use k colors. If $k > \chi(G)$, not all of the k colors need be used for each valid coloring.

Example. The chromatic polynomial $P_G(k)$ of the complete graph K^3 is simply the number of permutations of k colors on 3 vertices. There are k ways of choosing the color of vertex a, $k - 1$ ways of choosing the color of an adjacent vertex, and $k - 2$ ways of choosing the color of the final vertex. Hence, $P_G(k) = k(k-1)(k-2)$. Note that $P_G(1) = P_G(2) = 0$, indicating that no colorings exist for those numbers of colors. $P_g(3) = 3 \times 2 \times 1 = 6$ and $P_G(4) = 4 \times 3 \times 2 = 24$, implying that six colorings exist on G which use three colors, and 24 colorings exist on G which use four colors (even though not all four colors are used in each coloring). An example is shown for $k = 3$.

The chromatic polynomial has interesting recurrences. We first require some additional definitions.

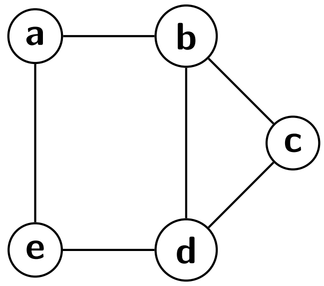


The 6 3-colorings of K^3.

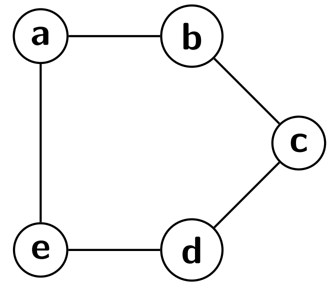
Definition. A **deletion** $G - e$ of a graph G is the graph G with the edge e removed.

Definition. A **contraction** G/e of a graph G is the graph with the edge $e = \{x, y\}$ contracted to a point. That is, we make the two separate vertices x and y into a single vertex z and connect z to the original neighbors of x and the original neighbors of y.

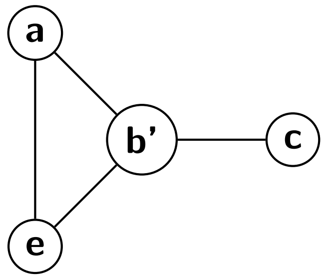
Example. A graph G with an edge e labelled, then the deletion $G-e$ then the contraction G/e.



The graph G.



G with the edge $\{b,d\}$ deleted.



G with the edge $\{b,d\}$ contracted to a single point b'.

Finally, we present an interesting theorem, which we will explore in parts (b), (c), and (d):

Theorem. *For any graph G, $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$.*

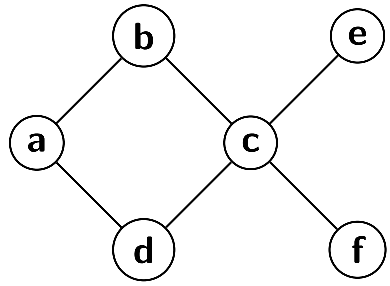
Note: In your solutions, you may leave polynomials in factorized form.

4. (a) Find the chromatic polynomial of the following graphs. Briefly justify your answers. You **may not** use the recurrence $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$ for your solutions in 4(a).
   i. **[2]** The complete graph K^n.
   ii. **[2]** An arbitrary tree of order n. A **tree** is a connected graph with no cycles.
   iii. **[3]** A cycle of order 4.

   (b) For any graph G, $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$. We will explore this recurrence in the following problems.
   i. **[2]** The recurrence relation holds for fixed k. Let G be the graph described in the example above, and choose edge $e = \{b,d\}$. Find $P_G(3)$ by computing $P_{(G-e)}(3)$ and $P_{(G/e)}(3)$.
   ii. **[2]** Let G be the graph described in the example above. Find the chromatic polynomial $P_G(k)$.
   iii. **[3]** For the graph H below, find $P_H(3)$.
   iv. **[3]** Find the chromatic polynomial $P_H(k)$.
   v. **[6]** Prove that the chromatic polynomial of a cycle C_n is

$$P_{C_n}(k) = (k-1)^n + (k-1)(-1)^n$$

.



A graph H.

vi. [**8**] Prove the recurrence $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$.

(c) In the next two problems, we explore the structure of the chromatic polynomial.

i. [**6**] Prove that for a graph G of order n, the chromatic polynomial $P_G(k)$ has degree n and leading coefficient 1. The degree of a polynomial is the highest power of k present in the polynomial. (Hint: Is there a way to apply our recurrence?)

ii. [**8**] Prove that for a graph G of order n with m edges, the coefficient of k^{n-1} in $P_G(k)$ is $-m$.

Acyclic Orientations

Up until now, we have been discussing graphs with undirected edges. Undirected edges usually symbolize connections or relationships. However, by assigning a direction to each edge, we can form a directed graph. In contrast to undirected edges, directed edges usually symbolize flows or transformations.

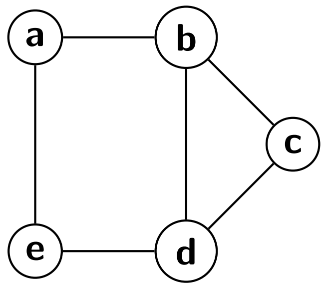
In this section, we prove a very nonintuitive connection between special orientations of graphs (called *acyclic orientations*) and the chromatic polynomial described in the previous section.

Definition. An **orientation** of a graph $G = (V, E)$ is the same graph $G = (V, E)$ with two additional maps, init : $E \to V$ and ter : $E \to V$ designating the **initial vertex** and **terminal vertex** of every edge e in E. If an edge $e = \{x, y\}$ is in E, then $\{\text{init}(e), \text{ter}(e)\} = \{x, y\} = \{y, x\} = e$.

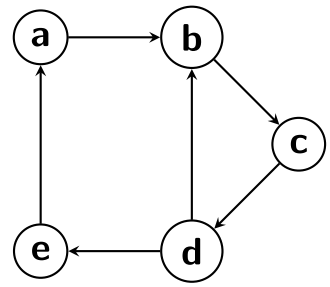
Definition. In an oriented graph, a **directed path** is a path $v_1v_2...v_k$ with edges v_iv_{i+1} and the additional requirement that $\text{init}(v_iv_{i+1}) = v_i$ and $\text{ter}(v_iv_{i+1}) = v_{i+1}$ for all $i = 1, 2, ..., k-1$. A **directed cycle** is a regular cycle with this additional requirement. Finally, an orientation is **acyclic** if it contains no directed cycles.

For a graph G, we denote the number of acyclic orientations of G (suggestively) as $\overline{\chi}(G)$.

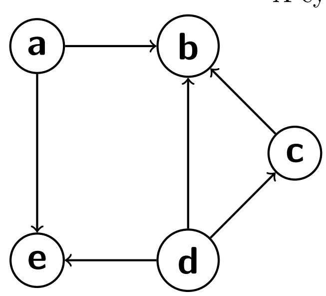
Example.



A graph G.



A cyclic orientation of G.



An acyclic orientation of G.

Finally, we present two interesting theorems regarding acyclic orientations, which we will explore in the following set of problems:

Theorem. *For any graph G, $\overline{\chi}(G) = \overline{\chi}(G - e) + \overline{\chi}(G/e)$.*

Theorem. *For any graph G, $\overline{\chi}(G) = |P_G(-1)|$.*

5. (a) Find the number of acyclic orientations for the following graphs. Briefly justify your answers. You **may not** use any of the theorems provided in this section for this part.
   i. **[2]** The complete graph K^3.
   ii. **[3]** The graph H (see problem 4.b.iii).
   iii. **[3]** The complete graph K^4.

   (b) Recall that acyclic orientations satisfy a useful recurrence relation: For any graph G, $\overline{\chi}(G) = \overline{\chi}(G - e) + \overline{\chi}(G/e)$. For the following problems, you **may not** use the equality $\overline{\chi}(G) = |P_G(-1)|$.
   i. **[2]** Let G be the cycle of degree 4. Show that the recurrence relation holds by computing $\overline{\chi}(G)$ using the recurrence relation.
   ii. **[4]** Let G be the graph described in the example above. Find $\overline{\chi}(G)$.
   iii. **[10]** Prove that for any edge e of G, $\overline{\chi}(G) = \overline{\chi}(G - e) + \overline{\chi}(G/e)$.

   (c) We now attempt to prove that $\overline{\chi}(G) = |P_G(-1)|$.
   i. **[3]** Using the theorem, find the number of acyclic orientations for K_n in terms of n, and show your work.
   ii. **[5]** Prove that $P_{G-e}(-1)$ and $P_{G/e}(-1)$ differ in sign. You may use without proof the fact that the coefficients of the chromatic polynomial alternate in sign.
   iii. **[8]** Using the recurrence relations for chromatic polynomials and acyclic orientations, prove that the number of acyclic orientations of G is $|P_G(-1)|$.