Introduction

This power round is about graph theory, and in particular, about graph colorings. In this case we refer to graphs as those with vertices and edges, and not those which are plots of functions. Usually, when we discuss colorings of a graph, we are concerned with assigning colors to the vertices or edges of the graph such that certain properties are satisfied. For example, we might be interested in colorings of graphs where no two adjacent vertices have the same color. This problem has numerous applications in scheduling and the development of compilers, and also has interesting applications to networking and parallelization of tasks. For example, if we need to complete a certain number of tasks which compete for resources, we can let vertices be tasks and let edges represent a conflict between two tasks for a certain resource. A coloring of this graph then represents a partition of the tasks into sets that can be accomplished simultaneously, with the number of colors used representing the number of instances we must assign tasks to resources.

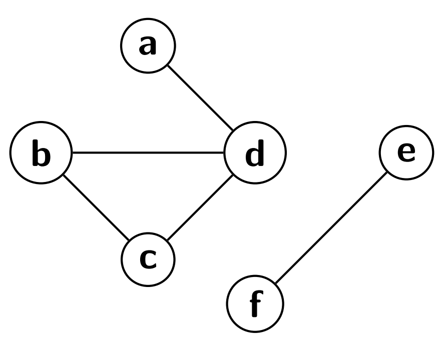
Graph colorings are a lot of fun and easily visualizable. We hope you enjoy working with these problems and that you leave with more experience and interest in the theory and applications of graph colorings.

Graph Basics

A caveat: In this section, we define a graph and many of its important features and properties. Although there are many terms and definitions in this section, they are meant to facilitate writing about graphs, and do not have much useful content in and of themselves. **We advise that you do not spend too much time reading and memorizing this first section, and instead treat it as a glossary for you to reference when in need of helpful language.**

Definition. A **graph** G is a pair (V, E) where V is a set of **vertices** and E is a set of **edges**. Each edge is a set $\{x, y\}$ where x and y are distinct vertices in V. We will usually denote such an edge as xy or yx . If such an edge exists, x and y are then said to be **adjacent** or **neighbors**. For a graph G, we refer to its set of vertices as $V(G)$ and its set of edges as $E(G)$. Additionally, we will define the number of vertices of G as its **order**. We disallow self-loops (edges connecting a vertex to itself) and multiple edges connecting the same pair of vertices.

Example.



A simple graph with 6 nodes and 5 edges.

Definition. A **complete** graph is one where every vertex is adjacent to every other vertex. We will denote the complete graph with n vertices as K^n.

Definition. The **degree** of a vertex is the number of edges incident on (or attached to) that vertex. For a vertex v in a graph G, we denote its degree with $d(v)$. The **maximum degree** (denoted by $\Delta(G)$) of a graph G is the largest degree of any vertex in G, and the **minimum degree** (denoted by $\delta(G)$) is the smallest degree of any vertex in G.

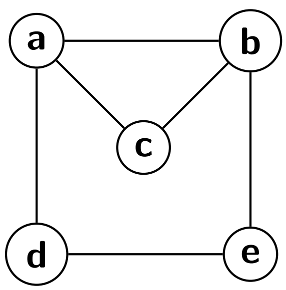
Example. The complete graph K_5 has $d(v) = \Delta(G) = \delta(G) = 4$ for every vertex v. In general, every vertex v in $V(K^n)$ has $d(v) = \Delta(G) = \delta(G) = n - 1$.

Definition. A **subgraph** G' of a graph $G = (V, E)$ is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. Informally, it is a graph obtained by taking G and selecting a set of edges and vertices such that the selection is still a valid graph. Notationally, we reuse the subset notation and say that $G' \subseteq G$ indicating that G' is a subgraph of G.

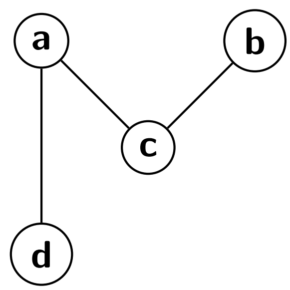
An **induced subgraph** G' is formed by selecting a subset V' of the vertices of G and then taking E' to be the set of edges between vertices in V' that are also in E.

For notational convenience, we will often refer to the graph G with vertex v removed (along with its associated edges) as $G - v$.

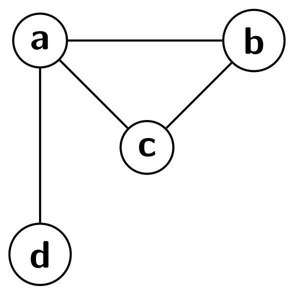
Example.



A graph G.



A subgraph of G.



The subgraph of G induced by nodes a, b, c, and d.

Definition. A **path** in a graph G is a route from one vertex to another, traveling only across edges of the graph. More precisely, it is the subgraph of G described by such a route. We will usually write a path from vertices v_0 to v_k as $P = v_0v_1v_2...v_k$. The length of a path is the number of edges in the path. A **cycle** with vertices $v_0...v_{k-1}$ is a path $v_0...v_{k-1}$ of length greater than 1, with the additional edge $v_{k-1}v_0$. The length of a cycle is the number of vertices (or edges) in the cycle. Note that paths and cycles do not have directions; they are simply graphs with vertices and edges.

Definition. A graph is **connected** if there exists a path from every vertex to every other vertex. A **connected component** or simply **component** of a graph G is an induced subgraph of G that is itself connected and to which no other vertices of G are connected, except the ones in the subgraph.

A note: for a few questions, we will ask you to draw a graph. Additionally, you may find it useful to reference graphs or parts of graphs when writing your proofs. When doing so, please draw the vertices and edges cleanly, and label specific vertices or edges if they are used in your proof. **Be advised that we will not tolerate messy graphs.**

1. (a) i. **[2]** Draw a connected graph G with 8 vertices such that $\Delta(G) = \delta(G) = 3$. (A graph with $\Delta(G) = \delta(G) = 3$ is called a *cubic* graph).
   ii. **[2]** Draw a connected graph G with more than 5 vertices such that $|V| = |E| + 1$. ($|S|$ denotes the number of elements in the set S.) This graph is an example of a special type of graph known as a **tree**.
   iii. **[2]** Draw a connected graph G with 4 vertices and 4 edges that is **bipartite** - that is, we can partition the vertices of G into two classes such that every edge of G has one endpoint in one class and one endpoint in the other.

   (b) **[4]** Prove that for any graph, there must be an even number of vertices of odd degree.

   (c) **[5]** Prove that a connected graph G is bipartite if and only if it contains no cycles of odd length.

   (d) **[6]** Prove that a graph G with $\delta(G) \geq 2$ has a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$. (Hint: Consider the longest path in G.)

   (e) Here, we explore the number of cycles in complete graphs.
   i. **[3]** Compute the number of cycles in K^4. Remember that a cycle is itself defined as a subgraph of G, and two graphs are equal if their edge sets and node sets are equal. You do not need to justify your answer.
   ii. **[7]** Derive a general formula for the number of cycles in K^n. For full credit, the formula should not be recursive (i.e. should not contain the number of cycles of K^m for $m \leq n$ as a term). You do not need to justify your answer.

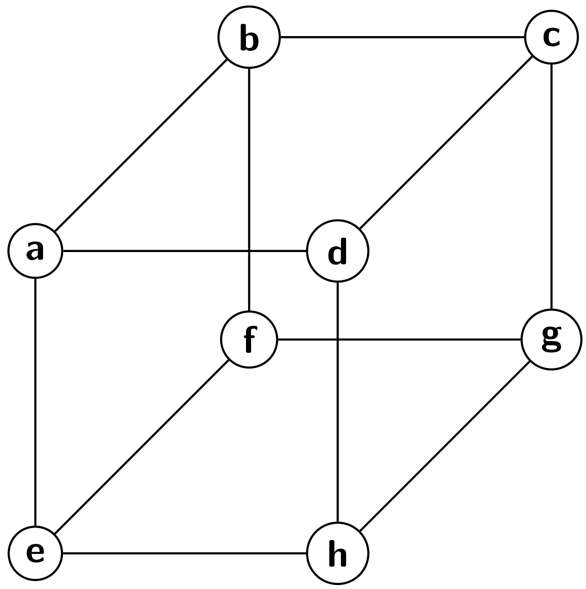
Solution to Problem 1:

(a) i. See graph below for a possible solution.
ii. There are many possible solutions, but any graph that lacks cycles and is formed of a single component is correct.
iii. See graph below.

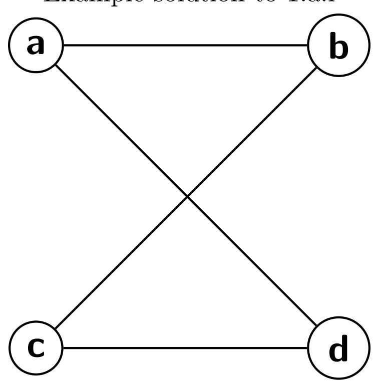
(b) Consider an arbitrary graph G. We begin by observing that the sum of all the degrees of G, $\sum_{v \in V} d(v)$, must be an even number because every edge contributes 2 to this sum. Thus, there must be an even number of v_i such that $d(v_i)$ is odd, otherwise $\sum_{v \in V} d(v)$ would itself be odd.

(c) If G is bipartite, label the two vertex sets 1 and 2. Then any cycle in G must have an even number of edges because every edge goes from 1 to 2 or vice versa, and so a path from a vertex v back to itself must have an edge from 2 to 1 for every edge from 1 to 2 in order to return to set 1 at the end of the cycle.

If G has no odd cycles, start with an arbitrary vertex v_0 and label it 1. Then label every vertex connected to it with 2. Continue labeling every vertex with the opposite color of those vertices it is connected to. Note that a vertex is labelled 1 if and only if there is a path of even length from v_0 leading to it, with a vertex being labelled 2 if and only if there is a path of odd length from v_0 leading to it. If we can do this without resulting in conflicts, then we can partition the vertices of G into two sets with every edge connecting a vertex labelled 1 with a vertex labelled 2, which is what we want.



Example solution to 1.a.i



Example solution to 1.a.iii

Assume a vertex v_k is connected to two vertices v_1 and v_2 of different colors, 1 and 2 respectively. For the vertex labelled 2, there is a path of odd length to the original vertex v_0. For the vertex labelled 1, there is a path of even length from v_0 leading to it. Thus the cycle formed by taking $v_0 ... v_1 v_k v_2 ... v_0$ has odd length, a contradiction.

(d) Consider the longest path in G, $P = v_1 ... v_k$. Then all of v_k's neighbors lie in P, otherwise we could append a neighbor of v_k to the end and come up with a longer path. Thus $k \geq d(v_k)$. Since $d(v_k) \geq \delta(G)$, $k \geq \delta(G)$ as desired. Now take the minimum i such that v_i is in P and $v_i v_k$ is an edge in G. Then $v_i v_{i+1} ... v_{k-1} v_k v_i$ is a cycle of length at most $\delta(G) + 1$.

(e) i. We can accomplish this by counting the number of 3-cycles and 4-cycles separately. Because K^4 is a complete graph, the number of 3-cycles is just the number of subsets of size 3 of the vertices, of which there are $\binom{4}{3} = 4$. Then the number of 4-cycles can be brute-force calculated to be 3, for a total of 7 cycles.

ii. We count the number of m-cycles for $3 \leq m \leq n$, and sum them all together. For an induced subgraph K^m of K^n, we label the vertices $1, ..., m$. Then there are $m!$ permutations of these vertex labels. Because a cycle $v_1 v_2 ... v_m v_1$ is equivalent to the cycle $v_2 ... v_m v_1 v_2$ and $v_3 ... v_m v_1 v_2 v_3$, etc. we divide the number of labels $m!$

by m to get $(m-1)!$, because the m permutations $123...m$, $23...m1$, $34...m12$, etc. must all correspond to the same cycle $v_1v_2...v_mv_1$. (Another way of deducing this is to fix the first element to be 1, and then see that there are $(m-1)!$ permutations for the remaining elements). Furthermore, because $v_1v_2...v_mv_1$ is equivalent to $v_mv_{m-1}...v_1v_m$, the permutations $123...m$ and $m...321$ must correspond to the same cycle, and we divide the number of permutations by 2 to get $(m-1)!/2$ cycles. Thus, the total number of m-cycles is the number of induced subgraphs K^m, which is just $\binom{n}{m}$, times the number of cycles using all the vertices in this subgraph, which is $(m-1)!/2$. Our formula is therefore:

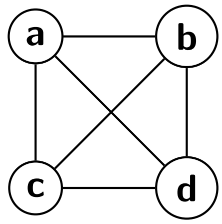
$$\sum_{m=3}^{n} \binom{n}{m} \frac{(m-1)!}{2}$$

Planar Graphs and the 5-Color Theorem

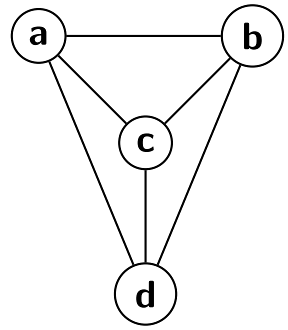
Planar graphs are a particularly interesting category of graphs that have some special coloring properties. One of the most famous theorems about planar graphs is the 4-color theorem, which states that any planar graph needs only 4 or fewer colors to be colored in such a way that adjacent vertices are different colors. It was finally proven in 1977 with the assistance of a computer - the problem was first reduced to a large but finite number of base cases, which the computer then verified. We begin with a few definitions.

Definition. A graph G is **planar** if it can be represented on a piece of paper such that no edges cross each other (or intersect with vertices other than their respective endpoints). See the example below.

Example.



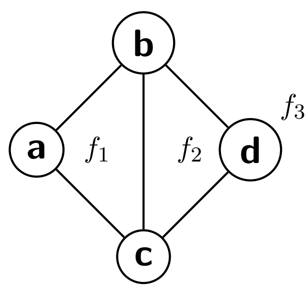
A planar graph drawn (incorrectly) with overlapping edges.



The same planar graph drawn without overlapping edges.

Definition. A **face** of a planar graph is a region in in the plane bounded by the edges and vertices of the graph (so long as the graph is drawn correctly as a planar graph with no intersecting edges). Note that the region "outside" of the graph is also a face. In the following picture, the faces are labeled f_1, f_2, and f_3 where the third face is the unbounded region external to the graph.

Example.



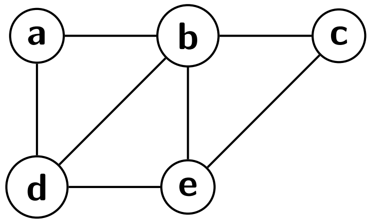
A simple graph.

Theorem (Euler's Formula). *If G is a connected planar graph with f faces, e edges, and v vertices, then $f - e + v = 2$.*

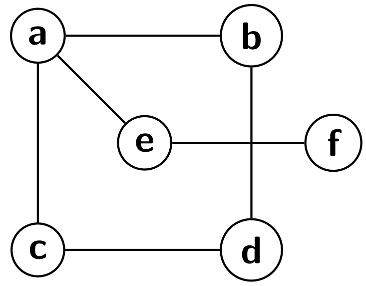
Definition. A **proper coloring** (or simply a **coloring**) of a graph G is a mapping from the vertices of G to colors $\{1, 2, ..., k\}$ such that adjacent vertices are different colors.

2. (a) i. [**2**] Confirm that Euler's Formula holds for graph (i) below, or explain why it doesn't. Show your work.
   ii. [**2**] Confirm that Euler's Formula holds for graph (ii) below, or explain why it doesn't. Show your work.
   iii. [**2**] Confirm that Euler's Formula holds for graph (iii) below, or explain why it doesn't. Show your work.

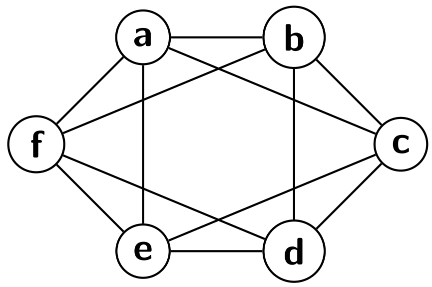
   (b) [**6**] Prove Euler's Formula for planar graphs. (Hint: Use induction on the number of edges. What might happen if you remove an edge from the graph? How will f, e and v be affected?)



Problem 2.a.i.



Problem 2.a.ii.



Problem 2.a.iii.

(c) The **5-color theorem** states that every planar graph has a coloring using a maximum of five colors. This is a relaxation of the 4-color theorem that can be proven with less casework. You **may not** use the 4-color theorem to prove any parts in this section.

i. **[5]** Use Euler's Formula to prove the following: A planar graph with $v \geq 3$ vertices has at most $3v - 6$ edges.

ii. **[5]** Use this to prove that a planar graph must have a vertex of degree ≤ 5.

iii. **[10]** Consider a counterexample to the theorem - a vertex with exactly five neighbors all of which are different colors $\{1, ..., 5\}$. If the vertices are colored from 1 to 5 clockwise, show that vertices colored 2 and 4 cannot be connected by a path containing vertices colored only 2 and 4. Use this fact to complete the proof. (Hint: Start by using induction on the number of vertices.)

Solution to Problem 2:

(a) i. $f = 4$, $e = 7$, and $v = 5$, hence $4 - 7 + 5 = 2$.

ii. Even though it looks like there are 3 faces, there are actually only 2. $f = 2$, $e = 6$, $v = 6$ so $2 - 6 + 6 = 2$.

iii. Interestingly, there is indeed a way to draw this as a planar graph! It can be done by e.g. redrawing fb, bd, and df to the outside of the outer hexagon. Then $f = 8$, $e = 12$, $v = 6$ and $8 - 12 + 6 = 2$.

(b) We induct on the number of edges. For $|E| = 1$, $f = 1$, $v = 2$, and $e = 1$ so $f - e + v = 2$. Now assume the theorem holds for all graphs with $|E| \leq n - 1$. We seek to show that it holds for $|E| = n$.

Consider G with $|E| = n$. Remove an edge v_1v_2. If it does not disconnect the graph, there must be a path from v_1 to v_2 that does not include v_1v_2. This would separate the plane into at least one face inside the cycle $v_1...v_2v_1$ and at least one face outside the cycle, which would then be combined by removing v_1v_2. Hence, $f - e - v = 2$ still holds for this case.

If removing v_1v_2 disconnects the graph, then each remaining component has fewer than n edges, and thus Euler's formula holds for each one independently: $f_1 - e_1 + v_1 = 2$ and $f_2 - e_2 + v_1 = 2$, with $e_1 + e_2 = n - 1$. Thus, $(f_1 + f_2) - (e_1 + e_2) + (v_1 + v_2) = 4$. But both f_1 and f_2 count the large external face that is the rest of the plane, so f for the original graph is actually $f_1 + f_2 - 1$. Therefore, $f - e - v$ for the original graph is given by

$$(f_1 + f_2 - 1) - (e_1 + e_2 + 1) + (v_1 + v_2) = 4 - 1 - 1 = 2$$

which proves the theorem.

(c) i. First, the graph with n vertices and the maximal number of edges has faces that are all triangles - that is, every face is bounded by exactly three edges, except for the outer containing face. If a face is bounded by more than 3 edges, it is a polygon which can be triangulated to add more edges without changing the number of vertices.

Consider such a graph G. Then because every edge divides two different faces and each face (except possibly the outer one) touches 3 edges we have $3f \leq 2e$. Plugging this in to Euler's formula yields $2 = f - e + v \leq 2e/3 - e + v$ which can be simplified to get $e \leq 3v - 6$.

ii. If the theorem is false, then the average degree of the vertices of d is greater than or equal to 6. The average degree can be expressed as $2e/v$, and then $2e/v \geq 6$ implies $e \geq 3v$, contradicting part (i) where $e \leq 3v - 6$.

iii. We induct on the number of vertices. Note that the theorem is clearly true for graphs with 1 to 5 vertices. Consider a planar graph of order $n \geq 6$ that cannot be 5-colored. Then there must be a vertex that is adjacent to at least 5 vertices all with different colors. By the preceding part, there must be a vertex with exactly 5 neighbors, all with different colors. Call the vertex v.

We will now show that it is possible to recolor the vertices of $H = G - v$ such that the neighbors of v are colorable with only 4 colors. Draw the graph without overlapping edges and color the neighbors of v 1 to 5 clockwise in the order of the angle at which their edges meet v. Label them $v_1, v_2, ...v_5$. Now consider the subgraph of H that consists only of vertices colored 1 or 3, denoted H_{13}. Then there must be a path in H from v_1 to v_3 of vertices alternating colors 1 and 3. If not, then v_1 and v_3 lie in different components of H_{13}, say $v_1 \in C_1$ and $v_3 \in C_3$ and we can swap colors 1 and 3 in C_3 to get a coloring of H where v_1 and v_3 are the same color (this is the crucial step).

Now, consider v_2 and v_4. We will show that they must lie in different components in H_{24}. Consider the path $H = v_1Pv_3$ such that P (possibly empty) connects v_1 and v_3. Then the cycle vv_1Pv_3v is a cycle of vertices not colored 2 or 4, and must enclose at least 1 face. However, because the edge vv_2 lies clockwise of vv_1 and counterclockwise of vv_3, v_2 must lie in the interior of this cycle (draw a picture!) and therefore must lie in a different component of H_{24} than v_2. Otherwise there would be an edge connecting vertices colored 2 and 4 that crosses an edge of vv_1Pv_3v.

Hence, v_4 can be recolored to be color 2 in $G - v$ and v can therefore be colored as well, as only 4 colors have been used in its 5 neighbors.

The Chromatic Number and Brook's Theorem

Now that we have some basic facts about graphs, it's time to examine one of the central concepts in graph coloring theory: the chromatic number.

Definition. The **chromatic number** of a graph G (denoted $\chi(G)$) is the minimum number of different colors required to color every vertex of G, such that each vertex is colored with exactly one color and adjacent vertices are different colors. We will sometimes refer to a graph G with $\chi(G) = n$ as an n**-colorable** graph.

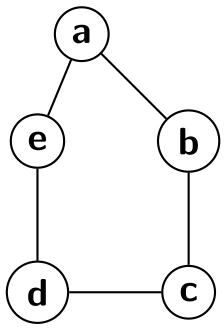
Example. For a complete graph K^n, $\chi(K^n) = n$ because every vertex is adjacent to every other vertex.

Often, the chromatic number is difficult to compute for an arbitrary graph. For example, it may depend on global properties (i.e. large-scale structures) of the graph that may be difficult to capture mathematically. However, there are some very useful bounds on the chromatic number of graphs that we can find using *local* properties of the graph (such as the degree of vertices). We examine some of the more common properties here.

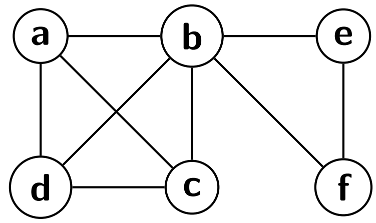
3. (a) i. **[2]** Find a 3-coloring of graph (i) below. Explain why its chromatic number is 3.

ii. **[2]** Find a 4-coloring of graph (ii) below. Explain briefly why its chromatic number is 4.

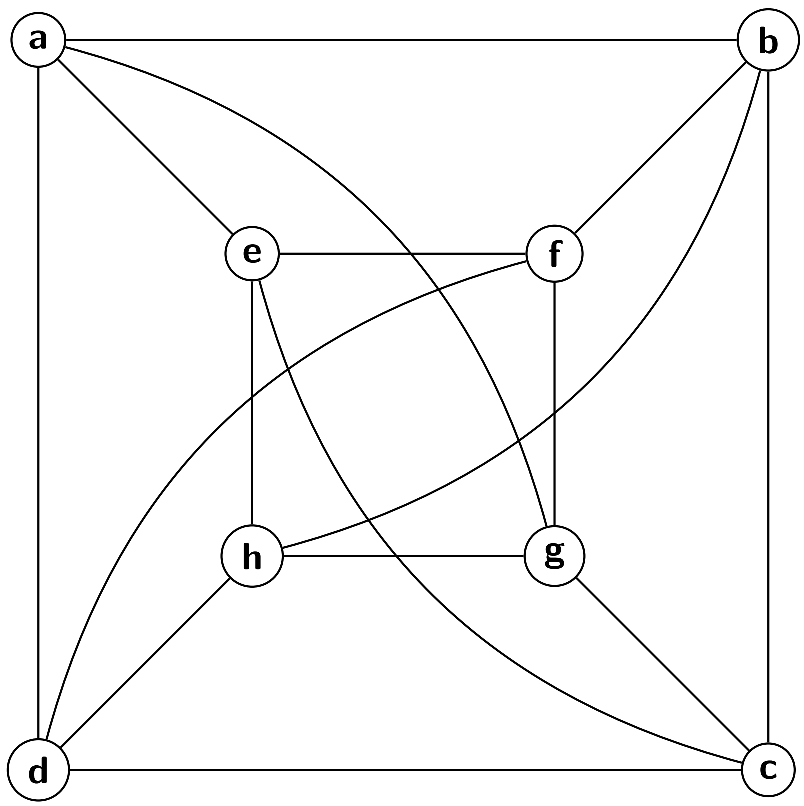
iii. **[2]** Find a 2-coloring of graph (iii) below. Explain briefly why its chromatic number is 2.



Problem 3.a.i.



Problem 3.a.ii.



Problem 3.a.iii.

(b) Sometimes, we can compute the chromatic number directly.

i. **[2]** Find the chromatic number of a connected bipartite graph. Justify your answer.

ii. **[3]** Find the chromatic number of an order $n+1$ **wheel graph** - a cycle of n vertices with an additional "central" vertex adjacent to all of the n vertices in the cycle, totaling $n+1$ vertices and $2n$ edges. Justify your answer.

(c) **[6]** Prove that any graph G with m edges satisfies $\chi(G) \leq 1/2 + \sqrt{2m + 1/4}$

(d) *Brooks' Theorem.* Prove that for any connected graph that is not a complete graph or an odd-length cycle, $\chi(G) \leq \Delta(G)$. This problem is cumulative - if you cannot prove a particular part, you may still use its results in later parts. We recommend drawing sketches and generally trying to visualize the concepts embodied in each section of this proof.

i. **[2]** Show why the theorem does not hold for complete graphs and odd-length cycles.

ii. **[2]** Show why the theorem holds for paths and even-length cycles (the only graphs with $\Delta(G) \leq 2$). This lets us assume that $\Delta(G) \geq 3$ for the following steps.

iii. **[5]** We now apply induction on the number of vertices. Assume all graphs H (except complete graphs and odd-length cycles) of order less than G satisfy $\chi(H) \leq \Delta(H)$. Consider a counterexample G such that $\chi(G) > \Delta(G)$. Remove a vertex v to create $H = G - v$. Show that v must have had exactly $\Delta(G)$ neighbors, all different colors. For future reference, we will call these vertices $v_1, v_2, ..., v_{\Delta(G)}$, colored 1, 2, ..., $\Delta(G)$ respectively.

iv. **[4]** Let $H_{i,j}$ be the induced subgraph of $G - v$ that contains exactly the vertices colored i and j, where $i \neq j$. Show that the vertices colored i and j that are adjacent to v must be in the same component $C_{i,j}$ of $H_{i,j}$.

v. **[6]** Note that v_i must have $\Delta(G) - 1$ neighbors in $G - v$, colored with $\Delta(G) - 1$ different colors, otherwise we could recolor v_i, leading to a contradiction with part iii. Now show that $C_{i,j}$ must be a path of vertices of alternating colors i and j from v_i to v_j. (Hint: Assume for the purpose of contradiction that there is a vertex u colored j on $C_{i,j}$ with three neighbors colored i (or colored i with three neighbors colored j), and show that it is then possible to recolor u).

vi. **[6]** Show that for distinct i, j, and k, $C_{i,j}$ and $C_{i,k}$ can only meet at v_i.

vii. **[10]** Show why the neighbors v_i of v cannot all be pairwise adjacent. Now complete the proof by finding a recoloring of $G - v$ that results in a contradiction of the fact proved in the previous part. (Hint: Consider a neighbor u of v_i that is not a neighbor of v. Can you recolor $C_{i,k}$ for some k in such a way that forces u to be a member of two different paths between neighbors of v?)

Solution to Problem 3:

(a) i. The chromatic number is three because it is an odd cycle.

ii. The chromatic number is at least four because the graph contains a subgraph that is K^4 By inspection, the remaining vertices can be colored using no more colors.

iii. The graph is bipartite with a, c, f, h as one group and b, d, e, g as the other.

(b) i. The chromatic number of a connected bipartite graph is 2. This is because the two bipartite classes can be colored different colors and every edge will automatically connect two vertex of different colors.

ii. The chromatic number of a wheel graph is 3 if n is even, and 4 if n is odd. If n is even, we color the cycle with 2 colors (alternating) and then color the central vertex the third color. If n is odd, we color the cycle with 3 colors, and then color the central vertex the fourth color.

(c) If the chromatic number is $\chi(G) = k$, then there must be at least one edge between all $k(k-1)/2$ pairs of color classes (i.e. for every pair of colors i and j, there must be an edge connecting a vertex colored i to a vertex colored j), otherwise we could obtain a coloring with fewer colors. Thus, $m \geq k(k-1)/2$. Solving this quadratic inequality (for example, by completing the square) for k yields the solution.

(d) i. Since $\Delta(G) = 2$ for odd length cycles G, we will attempt to find a 2-coloring of G. For odd length cycles $v_1....v_n$, n odd, v_n adjacent to v_1, we fix the color of v_1 as 1. Then all the even vertices $v_2, v_4, ...$ will need to be colored 2, and the odd vertices will need to be colored 1. But v_n is an odd vertex and is adjacent to 1, so this is impossible. For complete graphs K^n, we already have that $\chi(K^n) = n$. But $\Delta(K^n) = n-1$, so the theorem also does not hold in this case.

ii. Following a similar argument as in part i, we can fix the color of v_1 as 1 and then color the even vertices 2. In this way, we can always obtain a valid coloring of an even cycle or a path.

iii. By the inductive hypothesis, $G - v$ satisfies $\chi(G - v) \leq \Delta(G - v) \leq \Delta(G) = \Delta$. Therefore, we can Δ-color the vertices of $G - v$. Then, if not all the colors $1, ..., \Delta$ are used on the neighbors of v in $G - v$, we can add v back to $G - v$ and color it one of the unused colors, obtaining a Δ-coloring of G and showing that $\chi(G) \leq \Delta$. This would be fine, except that we assumed that G was a counterexample to the theorem (i.e. G cannot be Δ-colored). Hence, for G to be a counterexample, all the colors $1, ..., \Delta$ must be used on the neighbors of v. Therefore $d(v) \geq \Delta$. But since $d(v) \leq \Delta$ by definition of Δ, $d(v) = \Delta$ and v has exactly Δ neighbors, each colored a different color.

iv. Let v_i and v_j be the two neighbors of v that are colored i and j respectively. To prove that v_i and v_j must be in the same component of $H_{i,j}$ note that if they were in different components, we could swap colors i and j in one of those components to obtain a Δ coloring of $G - v$ that now colors v_i and v_j the same color! By part iii, this cannot happen, and so v_i and v_j must be in the same component.

v. Consider a vertex u not adjacent to v_i or v_j that is colored i with at least three neighbors colored j, or colored j with at least three neighbors colored j. Then it has at most $\Delta - 3$ remaining neighbors, and there remain $\Delta - 2$ colors to color them with. After coloring the neighbors of u, then there must be at least one unused color that is NOT i or j, so we can recolor u something other than i or j. Taking the first such u on the path from v_i to v_j, we can recolor this u to break $C_{i,j}$ into two components, one containing v_i and one containing v_j. This contradicts part iv, therefore $C_{i,j}$ must be just a path of vertices of alternating colors i and j from v_i to v_j.

vi. If $C_{i,j}$ and $C_{i,k}$ intersect at a vertex $u \neq v_i$, then u (which must be colored i) has two neighbors colored j and two neighbors colored k. Then u has $\Delta - 4$ remaining neighbors and $\Delta - 3$ colors remain to color them with - hence we have an extra color that we could use to recolor u, thus splitting the paths $C_{i,j}$ and $C_{i,k}$, leading to a contradiction.

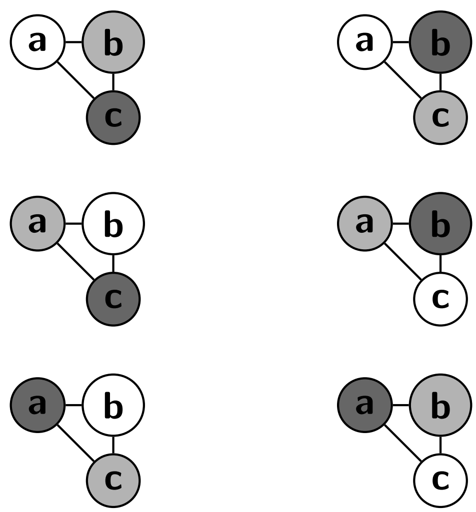
vii. If all Δ of the neighbors of v are pairwise adjacent, then they, along with v, form the complete graph $K^{\Delta+1}$ and there is nothing to prove. Now without loss of generality, assume that v_1 and v_2 are not neighbors, and that there exists some other vertex u colored 2 that is adjacent to v_1. Now swap the colors 1 and 3 in $C_{1,3}$. This gives us a new, equally valid coloring of $G - v$. Now consider the new components $C'_{1,2}$ and $C'_{2,3}$. Because v_1 is now colored 3, and u is adjacent to v_1 and is colored 2, u must be on $C'_{2,3}$ (from the problem statement of part v, u is the unique neighbor of v_1 that is colored 2). Similarly, because the colors of $C_{1,2}$ were not interchanged, the path $v_2...u$ is still part of $C'_{1,2}$. But then u is a part of both $C'_{1,2}$ and $C'_{2,3}$, and is not v_2, so we have a contradiction. This finishes the proof of Brooks' Theorem.

The Chromatic Polynomial

Now we will discuss a generalization of the chromatic number of a graph known as the chromatic polynomial.

Definition. The **chromatic polynomial** $P_G(k)$ of a graph G is a polynomial such that for each k, $P_G(k)$ gives the number of colorings of G that use k colors. If $k > \chi(G)$, not all of the k colors need be used for each valid coloring.

Example. The chromatic polynomial $P_G(k)$ of the complete graph K^3 is simply the number of permutations of k colors on 3 vertices. There are k ways of choosing the color of vertex a, $k-1$ ways of choosing the color of an adjacent vertex, and $k-2$ ways of choosing the color of the final vertex. Hence, $P_G(k) = k(k-1)(k-2)$. Note that $P_G(1) = P_G(2) = 0$, indicating that no colorings exist for those numbers of colors. $P_g(3) = 3 \times 2 \times 1 = 6$ and $P_G(4) = 4 \times 3 \times 2 = 24$, implying that six colorings exist on G which use three colors, and 24 colorings exist on G which use four colors (even though not all four colors are used in each coloring). An example is shown for $k = 3$.



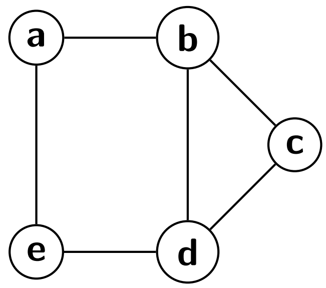
The 6 3-colorings of K^3.

The chromatic polynomial has interesting recurrences. We first require some additional definitions.

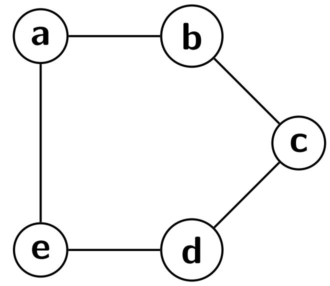
Definition. A **deletion** $G - e$ of a graph G is the graph G with the edge e removed.

Definition. A **contraction** G/e of a graph G is the graph with the edge $e = \{x, y\}$ contracted to a point. That is, we make the two separate vertices x and y into a single vertex z and connect z to the original neighbors of x and the original neighbors of y.

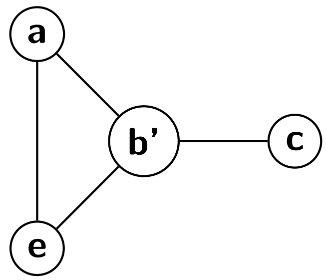
Example. A graph G with an edge e labelled, then the deletion $G - e$ then the contraction G/e.



The graph G.



G with the edge $\{b, d\}$ deleted.



G with the edge $\{b, d\}$ contracted to a single point b'.

Finally, we present an interesting theorem, which we will explore in parts (b), (c), and (d):

Theorem. *For any graph G, $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$.*

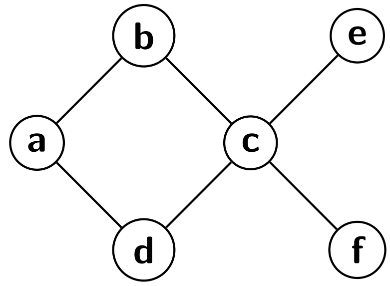
Note: In your solutions, you may leave polynomials in factorized form.

4. (a) Find the chromatic polynomial of the following graphs. Briefly justify your answers. You **may not** use the recurrence $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$ for your solutions in 4(a).
   i. [**2**] The complete graph K^n.
   ii. [**2**] An arbitrary tree of order n. A **tree** is a connected graph with no cycles.
   iii. [**3**] A cycle of order 4.

   (b) For any graph G, $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$. We will explore this recurrence in the following problems.
   i. [**2**] The recurrence relation holds for fixed k. Let G be the graph described in the example above, and choose edge $e = \{b, d\}$. Find $P_G(3)$ by computing $P_{(G-e)}(3)$ and $P_{(G/e)}(3)$.
   ii. [**2**] Let G be the graph described in the example above. Find the chromatic polynomial $P_G(k)$.
   iii. [**3**] For the graph H below, find $P_H(3)$.
   iv. [**3**] Find the chromatic polynomial $P_H(k)$.
   v. [**6**] Prove that the chromatic polynomial of a cycle C_n is

$$P_{C_n}(k) = (k-1)^n + (k-1)(-1)^n$$

.



A graph H.

vi. [**8**] Prove the recurrence $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$.

(c) In the next two problems, we explore the structure of the chromatic polynomial.

i. [**6**] Prove that for a graph G of order n, the chromatic polynomial $P_G(k)$ has degree n and leading coefficient 1. The degree of a polynomial is the highest power of k present in the polynomial. (Hint: Is there a way to apply our recurrence?)

ii. [**8**] Prove that for a graph G of order n with m edges, the coefficient of k^{n-1} in $P_G(k)$ is $-m$.

Solution to Problem 4:

(a) i. Let k be the number of available colors. An arbitrary uncolored vertex in K^n can have one of k colors. A second arbitrary uncolored vertex can have one of $k-1$ colors. Generalizing over all vertices, the chromatic polynomial of possible colorings is $P_G(k) = k(k-1)(k-2)...(k-n+1) = \frac{k!}{(k-n)!}$.

ii. Let k be the number of available colors. We assign an arbitrary uncolored vertex v one of k colors. Next, suppose that we traverse the tree starting from v. Since the tree has no cycles, we are guaranteed that each uncolored vertex we encounter has exactly one colored neighbor. Hence, the remaining $n-1$ vertices can have one of $k-1$ colors. We conclude that the chromatic polynomial of possible colorings is $P_G(k) = k(k-1)^{n-1}$.

iii. Suppose that the vertices in order are a, b, c, d. Then a is adjacent to d. We can color a one of k colors. Next, there are two cases:

A. b and d are different colors. There are $(k-1)(k-2)$ ways to color b and d, since b and d must be colored differently from a. Finally, there are $k-2$ ways to color c, since c must be colored differently from b and d.

B. b and d are the same color. There are $k-1$ ways to color b and d together. Next, there are $k-1$ ways to color c, since c must be colored differently from b and d.

Thus, the chromatic polynomial is

$$\begin{aligned} P_G(k) &= k(k-1)(k-2)^2 + k(k-1)(k-1) \\ &= k(k-1)(k^2 - 4k + 4 - (k-1)) \\ &= k(k-1)(k^2 - 5k + 5) \end{aligned}$$

(b) i. $G - e$ has $3 * 2^3 = 24$ possible colorings using 3 colors, and G/e has $3 * 2 * 2 = 12$ possible colorings. Hence, $P_G(3) = P_{(G-e)}(3) - P_{(G/e)}(3) = 24 - 12 = 12$.

ii. We observe that $P_{(G-e)}(k) = k(k-1)^3(k-2)$, while $P_{(G/e)}(k) = k(k-1)^2(k-2)$. Hence, $P_H(k) = k(k-1)^3(k-2) - k(k-1)^2(k-2) = k(k-1)^2(k-2)((k-1)-1) = k(k-1)^2(k-2)^2$.

iii. Suppose we let $e = \{c, d\}$. $H - e$ has $3 * 2^5 = 96$ possible colorings using 3 colors, and H/e has $3 * 2^3 = 24$ possible colorings using 3 colors. Hence, $P_H(3) = 96 - 24 = 72$.

iv. We observe that $P_{(H-e)}(k) = k(k-1)^5$, while $P_{(H/e)}(k) = k(k-1)^3(k-2)$. Hence, $P_H(k) = k(k-1)^5 - k(k-1)^3(k-2) = k(k-1)^3((k-1)^2 - (k-2)) = k(k-1)^3(k^2 - 3k + 3)$.

v. We proceed by induction on n. When $n = 2$, $P_{C_2}(k) = (k-1)^2 + (k-1) = k(k-1)$, which clearly holds. When $n = 3$, $P_{C_3}(k) = (k-1)^3 - (k-1) = (k-1)((k-1)^2 - 1) = k(k-1)(k-2)$, which again clearly holds. Next, suppose that $P_{C_n}(k) = (k-1)^n + (k-1)(-1)^n$ for all $n' < n$. Let e be an arbitrary edge of C_n. Note that $C_n - e$ forms a line graph and has a chromatic polynomial of $k(k-1)^{n-1}$. In addition, C_n/e is simply the graph C_{n-1}.
By the recurrence, we may write

$$\begin{aligned}
P_{C_n}(k) &= P_{C_n-e}(k) - P_{C_n/e}(k) \\
&= k(k-1)^{n-1} - ((k-1)^{n-1} + (k-1)(-1)^{n-1}) \\
&= k(k-1)^{n-1} - (k-1)^{n-1} - (k-1)(-1)^{n-1} \\
&= (k-1)^n + (k-1)(-1)^n
\end{aligned}$$

Thus we have proved the inductive step, and we conclude that $P_{C_n}(k) = (k-1)^n + (k-1)(-1)^n$ for all n.

vi. We prove the similar statement that $P_G(k) + P_{(G/e)}(k) = P_{(G-e)}(k)$. Consider the set S of all possible k-colorings of $G - e$; the size of this set is precisely $P_{(G-e)}(k)$. Consider an arbitrary k-coloring C of $G - e$, and suppose that the two endpoints of e in G are x and y. There are two cases:

A. x and y are different colors. Then C is also a valid k-coloring of G. This is because for all edges $\{a, b\}$ in G, either ab is an edge in $G - e$, and a and b are colored differently with the coloring C, or ab is the edge xy, where x and y are colored differently.

B. x and y are the same color. Then C is also a valid k-coloring of G/e. This is because for all edges $\{a, b\}$ in G/e, ab is an edge in $G - e$, and a and b are colored differently with the coloring C.

Hence, each k-coloring of $G - e$ can be mapped to either a k-coloring of G or of G/e. We conclude that $P_G(k) + P_{(G/e)}(k) = P_{(G-e)}(k)$, and equivalently stated we have $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}$.

(c) i. We proceed by induction on the number of edges. For the base case, suppose that a graph G has order n and zero edges. Then each of the n vertices can be colored k ways, and $P_G(k) = k^n$, which clearly has degree n and leading coefficient 1.
For the inductive step, assume that $P_G(k)$ has degree n and leading coefficient 1 for all graphs G with $m' < m$ edges, and consider a graph G with m edges. Let e be an arbitrary edge of G. From the recurrence, we may write $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$. $G - e$ is a subgraph with n vertices and $m - 1$ edges, hence by the inductive hypothesis we may write $P_{(G-e)}(k) = k^n + Q(k)$ where $Q(k)$ has degree at most $n - 1$. Next, G/e is a subgraph with $n - 1$ vertices and $m - 1$ edges,

hence by the inductive hypothesis $P_{(G/e)}(k)$ has degree $n-1$ and leading coefficient 1. Thus, we may write

$$P_G(k) = k^n + Q(k) - P_{(G/e)}(k)$$

Since k^n is the monomial of $P_G(k)$ with the largest degree, we conclude that $P_G(k)$ has degree n and leading coefficient 1. Thus, we have proved the inductive hypothesis, and conclude that the theorem holds for all graphs with any number of vertices n and any number of edges m.

ii. Again, we proceed by induction on the number of edges. For the base case, suppose that a graph G has order n and zero edges. Then each of the n vertices can be colored k ways, and $P_G(k) = k^n$. The coefficient of k^{n-1} is 0, which is equivalent to the negative of the number of edges, which is 0.
For the inductive step, assume that the coefficient of k^{n-1} in $P_G(k)$ is $-m'$ for all $m' < m$, and consider a graph G with m edges. Let e be an arbitrary edge of G. From the recurrence, we may write $P_G(k) = P_{(G-e)}(k) - P_{(G/e)}(k)$. $G-e$ is a subgraph with n vertices and $m-1$ edges, hence by the inductive hypothesis and the result of the previous problem we may write $P_{(G-e)}(k) = k^n - (m-1)k^{n-1} + Q(k)$, where $Q(k)$ has degree at most $n-2$. Next, G/e is a subgraph with $n-1$ vertices and $m-1$ edges, hence by the inductive hypothesis and the result of the previous problem we may write $P_{(G/e)}(k) = k^{n-1} + R(k)$, where $R(k)$ has degree at most $n-2$. Thus, we may write

$$\begin{aligned} P_G(k) &= k^n - (m-1)k^{n-1} - k^{n-1} + Q(k) - R(k) \\ &= k^n - mk^{n-1} + Q(k) - R(k) \end{aligned}$$

We conclude that the coefficient of k^{n-1} is $-m$, proving the inductive hypothesis. Thus, the theorem holds for all graphs with any number of vertices n and any number of edges m.

Acyclic Orientations

Up until now, we have been discussing graphs with undirected edges. Undirected edges usually symbolize connections or relationships. However, by assigning a direction to each edge, we can form a directed graph. In contrast to undirected edges, directed edges usually symbolize flows or transformations.

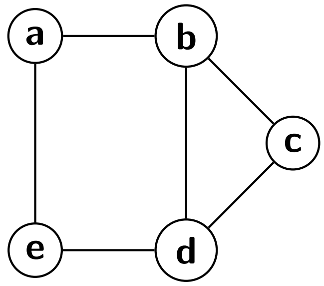
In this section, we prove a very nonintuitive connection between special orientations of graphs (called *acyclic orientations*) and the chromatic polynomial described in the previous section.

Definition. An **orientation** of a graph $G = (V, E)$ is the same graph $G = (V, E)$ with two additional maps, $\text{init} : E \to V$ and $\text{ter} : E \to V$ designating the **initial vertex** and **terminal vertex** of every edge e in E. If an edge $e = \{x, y\}$ is in E, then $\{\text{init}(e), \text{ter}(e)\} = \{x, y\} = \{y, x\} = e$.

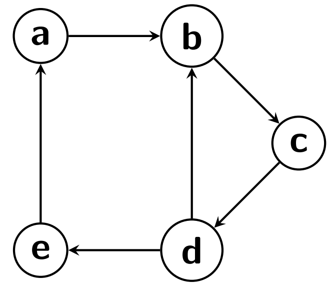
Definition. In an oriented graph, a **directed path** is a path $v_1v_2...v_k$ with edges v_iv_{i+1} and the additional requirement that $\text{init}(v_iv_{i+1}) = v_i$ and $\text{ter}(v_iv_{i+1}) = v_{i+1}$ for all $i = 1, 2, ..., k-1$. A **directed cycle** is a regular cycle with this additional requirement. Finally, an orientation is **acyclic** if it contains no directed cycles.

For a graph G, we denote the number of acyclic orientations of G (suggestively) as $\overline{\chi}(G)$.

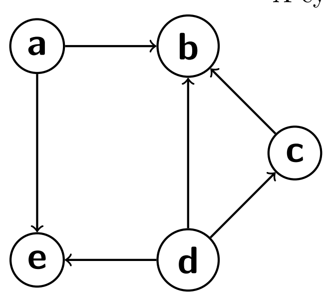
Example.



A graph G.



A cyclic orientation of G.



An acyclic orientation of G.

Finally, we present two interesting theorems regarding acyclic orientations, which we will explore in the following set of problems:

Theorem. *For any graph* G, $\overline{\chi}(G) = \overline{\chi}(G-e) + \overline{\chi}(G/e)$.

Theorem. *For any graph* G, $\overline{\chi}(G) = |P_G(-1)|$.

5. (a) Find the number of acyclic orientations for the following graphs. Briefly justify your answers. You **may not** use any of the theorems provided in this section for this part.
   - i. [**2**] The complete graph K^3.
   - ii. [**3**] The graph H (see problem 4.b.iii).
   - iii. [**3**] The complete graph K^4.

   (b) Recall that acyclic orientations satisfy a useful recurrence relation: For any graph G, $\overline{\chi}(G) = \overline{\chi}(G-e) + \overline{\chi}(G/e)$. For the following problems, you **may not** use the equality $\overline{\chi}(G) = |P_G(-1)|$.
   - i. [**2**] Let G be the cycle of degree 4. Show that the recurrence relation holds by computing $\overline{\chi}(G)$ using the recurrence relation.
   - ii. [**4**] Let G be the graph described in the example above. Find $\overline{\chi}(G)$.
   - iii. [**10**] Prove that for any edge e of G, $\overline{\chi}(G) = \overline{\chi}(G-e) + \overline{\chi}(G/e)$.

   (c) We now attempt to prove that $\overline{\chi}(G) = |P_G(-1)|$.
   - i. [**3**] Using the theorem, find the number of acyclic orientations for K_n in terms of n, and show your work.
   - ii. [**5**] Prove that $P_{G-e}(-1)$ and $P_{G/e}(-1)$ differ in sign. You may use without proof the fact that the coefficients of the chromatic polynomial alternate in sign.

iii. [**8**] Using the recurrence relations for chromatic polynomials and acyclic orientations, prove that the number of acyclic orientations of G is $|P_G(-1)|$.

Solution to Problem 5:

(a) i. There are $2^3 = 8$ possible orientations of K^3, and exactly two ways for those orientations to be cyclic. Hence we are left with 6 acyclic orientations.

ii. Consider the cycle formed by the vertices a, b, c, d. There are 2^4 possible orientations of the edges between those vertices, with 2 orientations being cyclic. Hence, there are $16 - 2 = 14$ acyclic orientations within the cycle. Next, any acyclic orientation does not depend on the orientation of $\{c, e\}$ and $\{c, f\}$. Since these two edges can each have 2 different orientations, H has $14 * 2 * 2 = 56$ acyclic orientations.

iii. There are four ways to align the middle two edges. Next, consider the edge connecting the terminal vertices of the two middle edges. Then the direction of one other edge is fixed. Finally, there are three ways to direct the remaining two edges. Thus, there are a total of $4 * 2 * 3 = 24$ acyclic orientations

(b) i. Consider an arbitrary edge e of G. Since $G-e$ has no cycles, $G-e$ has $2^3 = 8$ acyclic orientations, since each edge can be oriented two ways. Next, G/e is simply K^3, which has 6 acyclic orientations. Hence $G - e$ has $8 + 6 = 14$ acyclic orientations.

ii. Note that $G - \{b, d\}$ is the cycle of degree 5. Since there are $2^5 = 32$ possible orientations of $G - \{b, d\}$, and two of the orientations are cyclic, $G - \{b, d\}$ has 30 acyclic orientations. Next, $G/\{b, d\}$ has double the number of acyclic orientations of K^3, since $\{b, c\}$ can be directed in two different ways. Hence $G/\{b, d\}$ has $2 * 6 = 12$ acyclic orientations. Hence, G has a total of $30 + 12 = 42$ acyclic orientations.

iii. Let the endpoints of e be x and y, and consider an acyclic orientation of $G - e$. There are two ways to direct e to obtain an orientation of G. Assume for the sake of contradiction that both orientations of e result in cyclic orientations. This means that in $G - e$, there exists a directed path from x to y and a directed path from y to x. Hence, there exists a cycle which includes x and y, which contradicts our assumption of an acyclic orientation.

Hence, for any acyclic orientation of $G-e$, there are either one or two corresponding acyclic orientations of G. Note that in the case where an acyclic orientation of $G-e$ corresponds to two acyclic orientations of G, no directed path exists from x to y or from y to x, as no cycles are formed which includes x and y if e is included. In this case, when x and y are contracted to a single point to form G/e, we have a valid acyclic orientation of G/e.

Thus, if both directions of e result in an acyclic orientation of G, this forms a 1 to 1 correspondence with an acyclic orientation of $G - e$ and G/e. If only one direction of e results in an acyclic orientation of G, this forms a 1 to 1 correspondence with an acyclic orientation of $G - e$. Thus, the sizes of the sets are equal, and $\overline{\chi}(G) = \overline{\chi}(G - e) + \overline{\chi}(G/e)$.

(c) i. Recall that the chromatic polynomial of K_n is $P_{K_n}(k) = k(k-1)...(k-n+1)$. Hence, the number of acyclic orientations is

$$\overline{\chi}(K_n) = |P_{K_n}(-1)| = |-1(-1-1)(-1-2)...(-1-n+1)| = n!$$

ii. From earlier in the round, we have that $P_{G-e}(k)$ is a polynomial of degree n, while $P_{G/e}(k)$ is a polynomial of degree $n-1$. Hence, for nonnegative integers a_i, a'_i we may write the chromatic polynomials as

$$P_{G-e}(k) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - \ldots$$
$$P_{G/e}(k) = x^{n-1} - a'_{n-2}x^{n-2} + a'_{n-3}x^{n-3} \ldots$$

Letting $k = -1$, we have

$$\begin{aligned} P_{G-e}(-1) &= (-1)^n - a_{n-1}(-1)^{n-1} + a_{n-2}(-1)^{n-2} - \ldots \\ &= (-1)^n(1 + a_{n-1} + a_{n-2} + \ldots) \\ P_{G/e}(-1) &= (-1)^{n-1} - a'_{n-2}(-1)^{n-2} + a'_{n-3}(-1)^{n-3} \ldots \\ &= (-1)^{n-1}(1 + a'_{n-2} + a'_{n-3} + \ldots) \end{aligned}$$

Since each of the a_i, a'_i are nonnegative, the sign of $P_{G-e}(-1)$ is $(-1)^n$, and the sign of $P_{G/e}(-1)$ is $(-1)^{n-1}$. Hence, the two values differ in sign.

iii. Let G be the graph of order n with no edges. Since each vertex can be colored independently, $P_G(k) = k^n$, hence $|P_G(-1)| = |(-1)^n| = 1$. The number of acyclic orientations is trivially 1, since there are no edges to direct. Thus, we have equality, and the statement holds.
Next, suppose that $|P_G(-1)| = \overline{\chi}(G)$ for all graphs G with $m' < m$ edges, and let G be a graph with m edges. Consider the graphs $G - e$ and G/e. Since both graphs have less than m edges, from the inductive hypothesis we have

$$\begin{aligned} \overline{\chi}(G - e) &= |P_{G-e}(-1)| \\ \overline{\chi}(G/e) &= |P_{G/e}(-1)| \end{aligned}$$

Summing the two equations, we have

$$\begin{aligned} \overline{\chi}(G - e) + \overline{\chi}(G/e) &= |P_{G-e}(-1)| + |P_{G/e}(-1)| \\ \overline{\chi}(G) &= |P_{G-e}(-1) - P_{G/e}(-1)| \\ \overline{\chi}(G) &= |P_G(-1)| \end{aligned}$$

where in the second step, we use the fact that $P_{G-e}(-1)$ and $-P_{G/e}(-1)$ have the same sign, as proved in part (i). Hence, we have proved the inductive hypothesis, and we conclude that for all graphs G we have $\overline{\chi}(G) = |P_G(-1)|$.