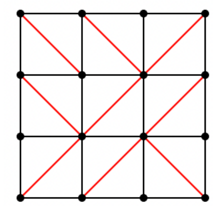
1. The boxes in the expression below are filled with the numbers 3, 4, 5, 6, and 7, so that each number is used exactly once, and the value of the expression is a prime number. Compute the value of the expression.

$$\square \times \square \times \square + \square \times \square$$

2. Compute the number of five letter words that can be formed from a, b, and c such that neither ab nor ac appears in the word. For example, $bcbaa$ would be valid but $aabcb$ would be invalid.

3. In a 3×3 grid, exactly one of the two possible diagonals is drawn in each of the nine cells. Compute the number of ways to choose the diagonals such that there is a continuous path along diagonals from the bottom-left vertex to the top-right vertex. One such grid is shown.



4. Professor Matrix Tessellation Laplace is thinking of two positive odd numbers (not necessarily distinct). Given that their product has exactly 15 positive divisors, compute the smallest possible sum of these two numbers.

5. Consider an axis-aligned 4 dimensional cube with side length 4, subdivided into 4^4 unit cubes. Cubey the penguin starts at a random unit cube. Each second, he can move to an adjacent unit cube along any one of the four axes. A fish is independently placed in a uniformly random unit cube (possibly the same one as Cubey's starting position) and the fish does not move (it's a fish out of water). Compute the expected time in seconds that Cubey will take to catch the fish, assuming he moves optimally to reach the fish as fast as possible.

6. Define the function $f(n)$ to be the remainder when $\binom{n}{20}$ is divided by 23. Find $f(20) + f(21) + \cdots + f(2023)$.

7. Professor Matrix Tessellation Laplace arranges all of the integers from 1 to 99999 (inclusive) in the order that maximizes the gigantic number formed by concatenating them. In his order, the number 2025 appears as the i-th integer from left to right. Find i.

   (Formally, he sets the permutation $\sigma = (\sigma_1, \ldots, \sigma_{99999})$ of $\{1, 2, \ldots, 99999\}$ such that the value of the concatenation $\overline{\sigma_1 \sigma_2 \ldots \sigma_{99999}}$ is maximized. Find the index i such that $\sigma_i = 2025$.)

8. For any permutation $\sigma = (a_1, a_2, \ldots, a_8)$ of the set $\{1, 2, \ldots, 8\}$ and some index $1 \leq k \leq 8$, define

$$L_k(\sigma) = \operatorname{lcm}(a_1, a_2, \ldots, a_k)$$

   where $\operatorname{lcm}(a_1, a_2, \ldots, a_k)$ denotes the least common multiple of $a_1, a_2, \ldots, a_k$. An index $2 \leq k \leq 8$ is called *increasing* if $L_k(\sigma) > L_{k-1}(\sigma)$. Let $T(\sigma)$ be the total number of increasing indices of σ. Compute the expected value of $T(\sigma)$ when σ is chosen uniformly at random among all permutations of $\{1, 2, \ldots, 8\}$.

9. Let α be the permutation of $\{0, 1, \ldots, 12\}$ satisfying $\alpha(0) = 0$, $\alpha(n) = n + 1$ for odd n, and $\alpha(n) = n - 1$ for positive even n. Let $f(n)$ be the unique polynomial of minimal degree with coefficients in $\{0, 1, \ldots, 12\}$ such that $13 \mid f(n) - \alpha(n)$ for all $n \in \{0, 1, \ldots, 12\}$. If $f(n)$ has degree d and can be expressed as $f(n) = \sum_{i=0}^{d} a_i n^i$, find $100a_d + a_1$.

10. Let s_0 be a square. One diagonal of s_0 is chosen, and a square s_1 is drawn with this diagonal as one of its sides. There are four possible such squares s_1; one is chosen out of these four uniformly at random. Choose s_2 in the same manner (so that it has a side which is a diagonal of s_1), and continue similarly with $s_3, s_4, \ldots$. Let $p(s_n)$ be the probability that square s_n shares an interior with s_0 (in other words, the area of their intersection is positive). Compute the value that $p(s_n)$ approaches as n approaches infinity.

TB. *This is an estimation question used for tiebreaking purposes. Ties on this test will be broken by absolute distance from the correct answer on this question.* A deck of cards consists of 52 cards in 13 ranks, with 4 cards in each rank. Let N be the number of ways to shuffle the deck so that any two cards with the same rank have no more than eight cards strictly between them (including other cards of the same rank).

Estimate the value of $\log_{10}(N)$ in the decimal form $abc.defgh$, where a, b, c, d, e, f, g, h are decimal digits each between 0 and 9, inclusive (leading zeros are allowed).