1. The boxes in the expression below are filled with the numbers 3, 4, 5, 6, and 7, so that each number is used exactly once, and the value of the expression is a prime number. Compute the value of the expression.

$$\square \times \square \times \square + \square \times \square$$

**Answer:** 107

**Solution:** The expression is a sum of two products. If the two products were to share any common factors, then the expression would be divisible by that common factor, which would make it not prime. Thus, 6 must be in the same product as 3, and also in the same product as 4. Therefore, the expression must be

$$3 \times 4 \times 6 + 5 \times 7 = 72 + 35 = \boxed{107}.$$

Indeed, 107 is a prime number.

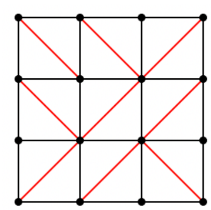
2. Compute the number of five letter words that can be formed from $a$, $b$, and $c$ such that neither $ab$ nor $ac$ appears in the word. For example, $bcbaa$ would be valid but $aabcb$ would be invalid.

**Answer:** 63

**Solution:** Once an $a$ appears, every letter to its right must also be an $a$. If $a$ first appears in position $i$, then there are $2^{i-1}$ possibilities, since each of the previous letters could be $b$ or $c$. If $a$ never appears, there are $2^5 = 32$ possibilities. This gives a total of

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 2^6 - 1 = \boxed{63}.$$

3. In a $3 \times 3$ grid, exactly one of the two possible diagonals is drawn in each of the nine cells. Compute the number of ways to choose the diagonals such that there is a continuous path along diagonals from the bottom-left vertex to the top-right vertex. One such grid is shown.



**Answer:** 79

**Solution:** First, observe that the bottom-left and top-right cells must be fixed and have their diagonals oriented to the right; otherwise, it would be impossible to start or end the path. Now we perform casework on the middle diagonal. If the middle diagonal faces right, then the path is automatically connected and we have $2^6$ possibilities for the remaining cells (their choices are independent, as the path is already determined). If the middle diagonal is oriented to the left, then to obtain a connected path, the diagonals in the remaining cells must form one of two specific patterns: either a path that goes up and right, up and left, up and right, down and right, up and right, or alternatively, a path that goes up and right, down and right, up and right, up and left, up and right. There are $2^3$ possibilities for the remaining cells after choosing one of these two paths, but we subtract 1 since we overcounted the possibility where both paths are created. In conclusion, there are $2^6 + 2^3 + 2^3 - 1 = \boxed{79}$ possible paths.

4. Professor Matrix Tessellation Laplace is thinking of two positive odd numbers (not necessarily distinct). Given that their product has exactly 15 positive divisors, compute the smallest possible sum of these two numbers.

**Answer:** 90

**Solution:** Let the two odd numbers be $a$ and $b$. We will minimize the product $ab$.
As the product has to also be odd, it cannot have 2 in its prime factorization. The number of divisors for a number $n$ with prime factorization $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ is $(e_1 + 1)(e_2 + 1)...(e_k + 1)$.
As $ab$ has 15 divisors, it is either of the form $p^{14}$ or $p^4 q^2$ for some primes $p$ and $q$.
The next smallest primes after 2 are 3 and 5. Clearly, $3^{14} > 3^4 5^2$ since $3^{10} > 5^2$, so the product $ab$ is minimized at $ab = 3^4 5^2 = 2025$. By AM-GM, the smallest sum occurs when $a = b = \sqrt{2025} = 45$, and $a + b = 90$. For any bigger product, the sum of $a + b$ is at least $2\sqrt{ab} > 2 \cdot 45 = 90$. So the smallest possible sum is in fact $\boxed{90}$.

5. Consider an axis-aligned 4 dimensional cube with side length 4, subdivided into $4^4$ unit cubes. Cubey the penguin starts at a random unit cube. Each second, he can move to an adjacent unit cube along any one of the four axes. A fish is independently placed in a uniformly random unit cube (possibly the same one as Cubey's starting position) and the fish does not move (it's a fish out of water). Compute the expected time in seconds that Cubey will take to catch the fish, assuming he moves optimally to reach the fish as fast as possible.

**Answer:** 5

**Solution:** Let Cubey's location be $(x_1, y_1, z_1, w_1)$ and the fish's location be $(x_2, y_2, z_2, w_2)$. It will take $d = |\ x_1 - x_2| + |\ y_1 - y_2| + |\ z_1 - z_2| + |\ w_1 - w_2|$ seconds for him to reach the fish. Then, we can use linearity of expectation and symmetry to write the expected distance as

$$\mathbb{E}[d] = \mathbb{E}[|\ x_1 - x_2|] + \mathbb{E}[|\ y_1 - y_2|] + \mathbb{E}[|\ z_1 - z_2|] + \mathbb{E}[|\ w_1 - w_2|] = 4 \cdot \mathbb{E}(|\ x_1 - x_2|)$$

If $x_1$ and $x_2$ can be any random integer from 1 to 4, then

$$\mathbb{E}[|\ x_1 - x_2|] = \frac{1}{16} \sum_{i_1=1}^{4} \sum_{i_2=1}^{4} |\ i_1 - i_2| = \frac{2}{16} \sum_{i_1=2}^{4} \sum_{i_2=1}^{i_1-1} (i_1 - i_2)$$

$$= \frac{1}{8} \sum_{i_1=2}^{4} (i_1 - 1)i_1 - \frac{i_1(i_1 - 1)}{2} = \frac{1}{8} \sum_{i_1=2}^{4} \frac{i_1^2 - i_1}{2} = \frac{1}{8}(1 + 3 + 6) = \frac{5}{4}.$$

This gives a final answer of $4 \cdot \frac{5}{4} = \boxed{5}$. An easier way to compute this sum is to manually write out the 16 possible pairs of $x_1, x_2$.

6. Define the function $f(n)$ to be the remainder when $\binom{n}{20}$ is divided by 23. Find $f(20) + f(21) + \cdots + f(2023)$.

**Answer:** 2024

**Solution:** Since 23 is a prime, by Wilson's Theorem, $22! \equiv -1 \equiv 22 \pmod{23} \implies 21! \equiv 1 \pmod{23}$. Again, since 23 is a prime, there exists a multiplicative inverse of $21 \pmod{23}$. As $21 \cdot 11 = 231 \equiv 1 \pmod{23}$, $20! \equiv \frac{1}{21} \equiv 11 \pmod{23}$, so $\frac{1}{20!} \equiv 21 \pmod{23}$.

Therefore,

$$\begin{aligned} f(n) &\equiv \binom{n}{20} \pmod{23} \\ &\equiv \frac{n!}{(n-20)!20!} \pmod{23} \\ &\equiv 21\frac{n!}{(n-20)!} \pmod{23} \end{aligned}$$

Note that $23 \nmid \frac{n!}{(n-20)!}$ only when $n \equiv 20, 21, 22 \pmod{23}$.

We have already done all the computation we need, $f(20) = 21 \cdot \frac{1}{21}$, $f(21) = 21 \cdot 1$, $f(22) \equiv 21 \cdot \frac{-1}{2} \equiv 1 \pmod{23}$

It follows that in every group of 23, we have a sum of 23. Thus, the answer is $88 \cdot 23 = \boxed{2024}$.

7. Professor Matrix Tessellation Laplace arranges all of the integers from 1 to 99999 (inclusive) in the order that maximizes the gigantic number formed by concatenating them. In his order, the number 2025 appears as the $i$-th integer from left to right. Find $i$.

(Formally, he sets the permutation $\sigma = (\sigma_1, ..., \sigma_{99999})$ of $\{1, 2, ..., 99999\}$ such that the value of the concatenation $\overline{\sigma_1\sigma_2...\sigma_{99999}}$ is maximized. Find the index $i$ such that $\sigma_i = 2025$.)

**Answer:** 88607

**Solution:** Denote the concatenation of two numbers $a$ and $b$ with $a \oplus b$. For any two numbers $a$ and $b$, let the number of digits of $a$ and $b$ be $m$ and $n$ respectively. $a \oplus b = a(10)^n + b$, $b \oplus a = b(10)^m + a$. As such,

$$a \oplus b > b \oplus a \iff a(10)^n + b > b(10)^m + a \iff \frac{a}{10^m - 1} > \frac{b}{10^n - 1}$$

In other words, for any integer $k$ with $d$ digits, we are ranking $k$ based on $\frac{k}{10^d-1}$ from largest to smallest. Since 2025 has a unique result for this expression, we can determine its position in the arrangement by calculating the number of integers from 1 to 99999 with this expression computed to be larger than 2025/9999. By case work: [label=]

1. 1 digit: from 2 to 9 $\implies$ 8 cases.
2. 2 digit: from 21 to 99 $\implies$ 79 cases.
3. 3 digit: from 203 to 999 $\implies$ 797 cases.
4. 4 digit: from 2026 to 9999 $\implies$ 7974 cases.
5. 5 digit: from 20252 to 99999 $\implies$ 79748 cases.

There are $8 + 79 + 797 + 7974 + 79748 = 88606$ cases in total, so 2025 is in the $\boxed{88607}$-th position.

8. For any permutation $\sigma = (a_1, a_2, ..., a_8)$ of the set $\{1, 2, ..., 8\}$ and some index $1 \leq k \leq 8$, define

$$L_k(\sigma) = \text{lcm}(a_1, a_2, ..., a_k)$$

where $\text{lcm}(a_1, a_2, ..., a_k)$ denotes the least common multiple of $a_1, a_2, ..., a_k$. An index $2 \leq k \leq 8$ is called *increasing* if $L_k(\sigma) > L_{k-1}(\sigma)$. Let $T(\sigma)$ be the total number of increasing indices of $\sigma$. Compute the expected value of $T(\sigma)$ when $\sigma$ is chosen uniformly at random among all permutations of $\{1, 2, ..., 8\}$.

**Answer:** $\frac{157}{40}$

**Solution:** Let $p(n)$ be the probability that $L$ strictly increases from $L_{k-1}$ to $L_k$ as the number $n$ appears in position $k$ among all permutations. By Linearity of Expectation, $\mathbb{E}[T(\sigma)] = \sum_{n=1}^{8} p(n) - \frac{7}{8}$, where $\frac{7}{8}$ corresponds to overcounting the step from $L_0$ to $L_1$, which occurs if $a_1$ is not 1, an event with probability $\frac{7}{8}$.

To find $p(n)$, note that an increase occurs when a higher exponent of a prime power is introduced. Clearly, $p(1) = 0$. Furthermore, as every number in $\{2, 3, 4, 5, 6, 7, 8\}$, barring 6, is some prime to some power, we can determine for each element $n \in \{2, 3, 4, 5, 7, 8\}$ a set $s(n)$ of elements that $lifts$ $n$. For $n$ to trigger an increase, $n$ must appear to the left of all its lifters in $s(n)$. This occurs with probability $\frac{1}{1+ \mid s(n) \mid}$ since $n$ must occur as the leftmost element in the ordered set $n \cup s(n)$. We have the following manual computation:

$n = 2$: The numbers must be multiple of 2, so $s(2) = \{4, 6, 8\}$ and $p(2) = \frac{1}{4}$

$n = 3$: This is only lifted by $\{6\}$. $p(3) = \frac{1}{2}$

$n = 4$: This is only lifted for $\{8\}$. $p(4) = \frac{1}{2}$

$n = 5$: No number lifts this number, so $p(5) = 1$

$n = 7$: No number lifts this number, so $p(7) = 1$

$n = 8$: No number lifts this number, so $p(8) = 1$

Finally, for $p(6)$, consider set $\{2, 3, 4, 6, 8\}$. For 6 to increase the lcm, The probability that 6 appears before 3 is $\frac{1}{2}$. The probability of 6 appearing before $\{2, 4, 8\}$ is $\frac{1}{4}$. The probability of 6 appearing before all of them is $\frac{1}{5}$. As such, $p(6) = \frac{1}{2} + \frac{1}{4} - \frac{1}{5} = \frac{11}{20}$ by principle of inclusion and exclusion. The final answer is

$$\begin{aligned}\mathbb{E}[T(\sigma)] &= \sum_{n=1}^{8} p(n) - \frac{7}{8} \\ &= \left(0 + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + 1 + \frac{11}{20} + 1 + 1\right) - \frac{7}{8} \\ &= \boxed{\frac{157}{40}}.\end{aligned}$$

9. Let $\alpha$ be the permutation of $\{0, 1, ..., 12\}$ satisfying $\alpha(0) = 0$, $\alpha(n) = n + 1$ for odd $n$, and $\alpha(n) = n - 1$ for positive even $n$. Let $f(n)$ be the unique polynomial of minimal degree with coefficients in $\{0, 1, ..., 12\}$ such that $13 \mid f(n) - \alpha(n)$ for all $n \in \{0, 1, ..., 12\}$. If $f(n)$ has degree $d$ and can be expressed as $f(n) = \sum_{i=0}^{d} a_i n^i$, find $100a_d + a_1$.

**Answer:** 608

**Solution:** We will outline a general method applicable to all functions modulo some prime $p$.
For any function $\theta : \{0, 1, ..., p - 1\} \to \{0, 1, ..., p - 1\}$, let

$$f_\theta(n) \equiv \sum_{i=1}^{p} \theta(i)(1 - (n - i)^{p-1})$$

with coefficients reduced to residues modulo $p$. $f_\theta$ is a polynomial with coefficients in $\{0, 1, ..., p - 1\}$ with $p \mid f(n) - f_\theta(n)$ for all $n \in \{0, 1, ..., p - 1\}$. $f_\theta$ is always is of degree at most $p - 1$. There are $p^p$ such polynomials and $p^p$ functions $\theta$, so $f_\theta$ is the unique polynomial with degree at most $p - 1$ corresponding to $\theta$. Therefore, $f_\theta$ is the minimal polynomial satisfying these conditions.

Applying this method to this problem, note that for $i \neq 0$, $\alpha(n) = n + (-1)^{n+1}$, and $\alpha(n) = 0$ for $n = 0$. From this,

$$\alpha(n) \equiv \sum_{i=1}^{12} (-1)^{i+1}(1 - (n - i)^{12})$$

First, we find $a_1$, which is the coefficient of $n$. We have

$$\begin{aligned} a_1 &\equiv 1 + \sum_{i=1}^{12} (-1)^{i+1}(-12(-i)^{11}) \\ &\equiv 1 + 12 \sum_{i=1}^{12} (-1)^{i+1} i^{11} \\ &\equiv 1 - \sum_{i=1}^{12} \frac{(-1)^{i+1}}{i} \end{aligned}$$

where $\frac{1}{i}$ denotes the multiplicative inverse of $i$ modulo 13. Note that the last line is due to Fermat's Little Theorem. As $i^{12} \equiv 1$ for all $i \neq 0$, $i^{11} \equiv \frac{1}{i}$ for all $i \neq 0$.
Finally, we can compute

$$
\begin{aligned}
&\sum_{i=1}^{12} \frac{(-1)^{i+1}}{i} \\
&\equiv \sum_{i=1}^{12} \frac{1}{i} - 2\left(\sum_{i=1}^{6} \frac{1}{2i}\right) \\
&\equiv \sum_{i=1}^{12} \frac{1}{i} - \sum_{i=1}^{6} \frac{1}{i} \\
&\equiv \sum_{i=7}^{12} \frac{1}{i} \\
&\equiv -\sum_{i=1}^{6} \frac{1}{i} \\
&\equiv -(1+7+9+10+8+11) \\
&\equiv 6
\end{aligned}
$$

And $a_1 \equiv 1 - 6 \equiv 8 \bmod 13$. It's clear that $d \leq 12$. We check that

$$a_{12} = \sum_{i=1}^{12} (-1)^{i+1}(-1) = 0$$

and

$$
\begin{aligned}
a_{11} &= \sum_{i=1}^{12} (-1)^{i+1}(-12(-i)) \equiv \sum_{i=1}^{12} (-1)^i i \\
&\equiv \sum_{j=1}^{6} (-1)^{2j-1}(2j-1) + (-1)^{2j}(2j) \equiv 6
\end{aligned}
$$

Therefore, $d = 11$ and $a_d = 6$ for final answer of $100(6) + 8 = \boxed{608}$.

10. Let $s_0$ be a square. One diagonal of $s_0$ is chosen, and a square $s_1$ is drawn with this diagonal as one of its sides. There are four possible such squares $s_1$; one is chosen out of these four uniformly at random. Choose $s_2$ in the same manner (so that it has a side which is a diagonal of $s_1$), and continue similarly with $s_3, s_4, \ldots$. Let $p(s_n)$ be the probability that square $s_n$ shares an interior with $s_0$ (in other words, the area of their intersection is positive). Compute the value that $p(s_n)$ approaches as $n$ approaches infinity.

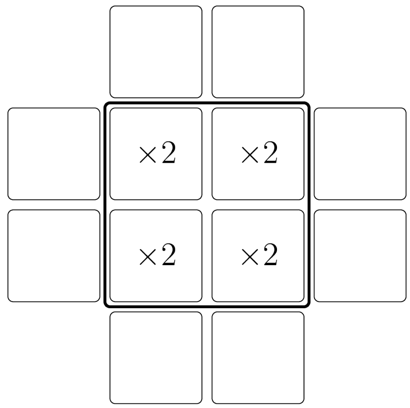
**Answer:** $\frac{5}{12}$

**Solution:** There are two intended solutions for this problem. We first present the simpler one:

SOLUTION ONE. In this solution we assume that the constraints of the problem are correct, namely that the sequence $p(s_1), p(s_2), \ldots$ indeed converges. Due to this, any infinite subsequence of this sequence must converge to the same value. Therefore, we can compute the limit of any infinite subsequence; we take the even-indexed values $p(s_2), p(s_4), \ldots$ so that if $p(s_n)$ is a member of this sequence, $s_n$ has parallel sides with $s_0$.

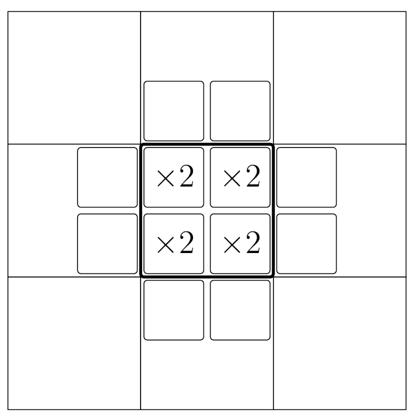
We also reframe the problem: instead of having $s_{i+1}$ be a square whose side is a diagonal of $s_i$, we let $s_{i+1}$ be a square whose diagonal is a side of $s_i$. We can see that gives a probability equivalent to the original process: by shifting all possible positions of $s_n$ to the same place, we note that $s_0, s_1, \ldots$ have taken the place of $s_n, s_{n-1}, \ldots$ and the process has been run in reverse.

Now, we can depict the possible values of $s_2$ as follows:



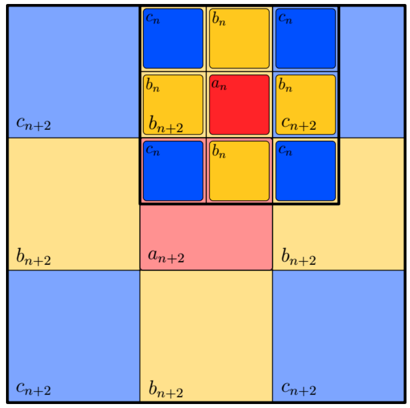

We see that there are 16 posssibilities, with 8 of them inside $s_0$ and 8 of them outside $s_0$. The next squares in the sequence $s_4, s_6, \ldots$ are generated by recursively enumerating this pattern for each of the smaller squares in the diagram.

Since the side length of $s_i$ halves each time $i$ is increased by 2, note that no point of $s_n$ will ever be outside of the boundary of a three by three grid of $s_0$'s:




Recall our goal is to find, in the limit, what proportion of the squares $s_n$ lie inside $s_0$. Note that the square $s_n$ will always either lie entirely within $s_0$ or have zero overlap with $s_0$. This is because when dividing each cell of our three by three grid into $2^n \times 2^n$ subcells, $s_n$ will always coincide exactly with one of these subcells. So, we analyze which cell of the large three by three grid $s_n$ lies in.

Suppose that when enumerating the pattern up to $s_n$, the copies of $s_n$ are distributed such that a proportion $a_n$ lie in the center cell, $b_n$ in an edge cell, and $c_n$ in a corner cell, where $a_n + b_n + c_n = 1$. Then, note that the pattern of the distribution of copies of $s_{n+2}$ can be attained by enumerating the pattern for $s_n$ on each of the copies of $s_2$:




This allows us to compute a recurrence relation for the sequences $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$. For each of the edge copies of $s_2$ (one of which is depicted above), we have that $\frac{1}{4}b_n + \frac{1}{4}c_n$ of its copies of $s_{n+2}$ end up in the large grid's center cell and $a_n + \frac{1}{2}b_n + \frac{1}{2}c_n$ of its copies end up in an edge cell. These contribute to $a_{n+2}$, $b_{n+2}$, and $c_{n+2}$ respectively.

For each of the center copies of $s_2$, we have that $a_n + \frac{1}{2}b_n + \frac{1}{4}c_n$ of its copies of $s_{n+2}$ end up in the large grid's center cell and $\frac{1}{2}b_n + \frac{1}{2}c_n$ of its copies end up in an edge cell.

Since there are as many edge copies of $s_2$ as center copies, we can simply take the averages of each of the contributions to achieve the equations

$$a_{n+2} = \frac{1}{2}a_n + \frac{3}{8}b_n + \frac{1}{4}c_n$$
$$b_{n+2} = \frac{1}{2}a_n + \frac{1}{2}b_n + \frac{1}{2}c_n.$$

Now, the convergent point must be at equilibrium, so we let $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ tend to $a$, $b$, and $c$ respectively as $i$ approaches infinity. Then we have that

$$\begin{cases} a = \frac{1}{2}a + \frac{3}{8}b + \frac{1}{4}c \\ b = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \\ a + b + c = 1 \end{cases}$$

which produces the solution $(a, b, c) = \left(\frac{5}{12}, \frac{1}{2}, \frac{1}{12}\right)$. Our desired quantity, which is the proportion of copies of $s_n$ that stay within the center cell of the three by three grid, is thus $\boxed{\frac{5}{12}}$.

SOLUTION TWO. If one wishes to show that $p$ does not oscillate between odd and even terms, there is an alternative more complicated solution using random variables and calculus. We first consider the following reverse process as in solution one. Beginning with some square $t_0$, for every square $t_n$, randomly select a side of $t_n$ and make that the diagonal of $t_{n+1}$. Let $p(t_n)$ be the probability that square $t_n$ shares an interior with $t_0$. We claim that $p(t_n) = p(s_n)$ for all $n$. This is due to the fact that for every sequence of square $s_0, \ldots, s_n$, if we scale and translate the squares such that $s_n$ maps to $t_0$, then $s_n, \ldots, s_0$ is a valid sequence of $t_0, \ldots, t_n$ with $t_0 = s_n$ and $t_n = s_0$ sharing an interior. Since there are $4^n$ total possible sequences in both $s_0, \ldots, s_n$ and $t_0, \ldots, t_n$, this must be a bijection, and therefore $p(t_n) = p(s_n)$

As such, we can instead compute $p(t_n)$ as $n$ approaches infinity. Furthermore, as the side length of square $t_n$ approaches 0 as $n$ approaches infinity, $p(t_n)$ is the probability that the center of square $t_n$ lies within square $t_0$.

Without loss of generality, let the starting square $t_1$ be centered at origin $(0, 0)$ with side length 2 with vertices on points $(2, 0), (0, 2), (-2, 0), (0, -2)$, and let $(x, y)$ be the coordinate of the center of the latest square $t_n$ as $n$ approaches infinity. We have the following observations:

- At every even step (from $t_{2k}$ to $t_{2k+1}$), with equal probability of $\frac{1}{4}$, $(x, y)$ can change by $\left(-\frac{1}{2^k}, 0\right), \left(\frac{1}{2^k}, 0\right), \left(0, -\frac{1}{2^k}\right), \left(0, \frac{1}{2^k}\right)$.
- At every odd step (from $t_{2n+1}$ to $t_{2n+2}$), $(x, y)$ can change by $\left(\pm\frac{1}{2} \cdot \frac{1}{2^k}, \pm\frac{1}{2} \cdot \frac{1}{2^k}\right)$.

Let us consider only the odd steps. Let $X_k$ and $Y_k$ be the random variables of changes of $x$ and $y$ coordinates of the center of the latest square at the $k$-th odd step, respectively, scaled by $2^k$. In other words, $(X_k, Y_k) = 2^k(x_{2k+2} - x_{2k+1}, y_{2k+2} - y_{2k+1})$. By the formula we obtained above, for any $k$, $X_k, Y_k$ each has $\frac{1}{2}$ probabiliy of being $-\frac{1}{2}$ and $\frac{1}{2}$ probability of being $+\frac{1}{2}$. Note that $X_k$ and $Y_k$ are independent. Let $X$ and $Y$ be random variables representing sums of changes in all odd steps from $k = 0$ to $\infty$. Then, $X = \sum_{k=0}^{\infty} \frac{X_k}{2^k}$ and $Y = \sum_{k=0}^{\infty} \frac{Y_k}{2^k}$, so $X$ and $Y$ are also independent.

Let ${X'}_k = X_k + \frac{1}{2}$, ${X'}_k$ is a random variable with $\frac{1}{2}$ chance of being 0 and $\frac{1}{2}$ chance of being 1.

$$X = \sum_{k=0}^{\infty} \frac{X_k}{2^k} = \sum_{k=0}^{\infty} \frac{{X'}_k}{2^k} - \sum_{k=0}^{\infty} \frac{\frac{1}{2}}{2^k} = \sum_{k=0}^{\infty} \frac{{X'}_k}{2^k} - 1.$$

Since $X_{(k)'}$ is a random variable with even likelihood of being 0 and 1, the sum $\sum_{k=0}^{\infty} \frac{{X'}_k}{2^k}$ can be considered as randomly choosing digits in the binary expansion for some real number between 0 and 2. This is uniform for all reals in this interval, so $X = U(0, 2) - 1 = U(-1, 1)$. By the same argument, $Y = U(-1, 1)$.

Let us now look at even steps. We similarly define $A_k$ and $B_k$ of $k$-th even step for all nonnegative integers $k$. However, note that $A_k$ and $B_k$ are not independent. We instead define random variables $U_k$ and $V_k$ such that $U_k = A_k + B_k$ and $V_k = A_k - B_k$, and $U_k, V_k$ are independent, each with $\frac{1}{2}$ chance of being $-1$ or 1. By similar argument scaled by 2, we conclude that $U = \sum_{k=0}^{\infty} \frac{U_k}{2^k}$ and $V = \sum_{k=0}^{\infty} \frac{V_k}{2^k}$ are uniform random variables on interval $(-2, 2)$. Let $(A, B)$ be the sum of changes of the even steps, so

$$A = \sum_{k=0}^{\infty} \frac{A_k}{2^k} = \sum_{k=0}^{\infty} \frac{\frac{U_k+V_k}{2}}{2^k} = \frac{U+V}{2},$$

and similarly

$$B = \frac{U-V}{2}.$$

$U$ and $V$ being uniform random variables on interval $(-2, 2)$ implies that for the even steps, the density of $(a, b)$ is uniform on a square with vertices at $(2, 0), (0, 2), (-2, 0), (0, -2)$.

Finally, we have random variables $X, Y, A, B$ where $(x, y)$ is the total change summed from all odd steps, $(a, b)$ is the total change summed from all even steps, and we are looking for the probability that $(x + a, y + b)$ is inside square $t_0$. Denote this probability by $p$. This is true if and only if $\mid x + a \mid \leq 1$ and $\mid y + b \mid \leq 1$. We may restrict the case to nonnegative $(a, b)$, as negative cases are equivalent by setting the corresponding $x$ or $y$ to be negative. Thus, fixing nonnegative $a, b \geq 0$, we have:

$$\max(-1, -1 - a) \leq x \leq \min(1, 1 - a)$$

$$\implies -1 \leq x \leq 1 - a$$

and

$$\max(-1, -1 - b) \leq y \leq \min(1, 1 - b)$$

$$\implies -1 \leq y \leq 1 - b.$$

Therefore, $p$ given $(a, b)$, which is the probability of $(x, y)$ being in this region, is

$$\frac{((1 - a) - (-1))((1 - b) - (-1))}{(2)(2)} = \frac{(2 - a)(2 - b)}{4}.$$

The nonnegative region of $(a, b)$ in the first quadrant is a right triangle with vertices at $(0, 0), (2, 0), (0, 2)$. Taking an integral gives us

$$p = \frac{\int_0^2 \int_0^{2-a} \frac{(2-a)(2-b)}{4} db\, da}{\frac{1}{2}(2)(2)} = \boxed{\frac{5}{12}}$$

TB. *This is an estimation question used for tiebreaking purposes. Ties on this test will be broken by absolute distance from the correct answer on this question.* A deck of cards consists of 52 cards in 13 ranks, with 4 cards in each rank. Let $N$ be the number of ways to shuffle the deck so that any two cards with the same rank have no more than eight cards strictly between them (including other cards of the same rank).

Estimate the value of $\log_{10}(N)$ in the decimal form $abc.defgh$, where $a, b, c, d, e, f, g, h$ are decimal digits each between 0 and 9, inclusive (leading zeros are allowed).

**Answer:** 44.22904190493972

**Solution:**