1. Ethan likes to build sandcastles on the beach. Let t, h, e, b, a, c be integers such that $b + e \neq 0$ and

$$\frac{t^h e}{b+e} = a - ch.$$

If $b = 0, c = 1, t = 2$, and $h = 3$, compute a.

Answer: 11

Solution: The left-hand side simplifies as

$$\frac{2^3 e}{0+e} = 8.$$

The right hand side simplifies as

$$a - 1(3) = a - 3.$$

Equating both sides, we find that $a = 8 + 3 = \boxed{11}$.

2. An isosceles triangle has legs of length $9! + 8!$, a base of length $2(9! - 8!)$, and a height of length $m \cdot 8!$ perpendicular to the base for some positive integer m. Compute m.

(Here, $n!$ represents the factorial of n, or the product of all positive integers up to n. For example, $3! = 3 \times 2 \times 1 = 6$ and $2! = 2 \times 1 = 2$.)

Answer: 6

Solution: Note that half of the base has length $9! - 8!$. Thus by the Pythagorean Theorem and difference of squares factorization between the halved base and the leg, we find that the height is

$$\sqrt{(9!+8!)^2 - (9!-8!)^2} = \sqrt{(9!^2 + 2 \cdot 9! \cdot 8! + 8!^2) - (9!^2 - 2 \cdot 9! \cdot 8! + 8!^2)} = \sqrt{2(9!)2(8!)}$$

$$= \sqrt{4 \cdot 9(8!)^2} = \sqrt{36} \cdot 8! = 6 \cdot 8!.$$

Thus, the answer is $m = \boxed{6}$.

Alternatively, one can scale everything down by 8! and the problem can be simplified to the case of a $3 - 4 - 5$ triangle.

3. Parker writes down all of the five-digit positive integers that use each of the digits 1, 2, 3, 4, and 5 exactly once, and also satisfy these two properties:

- The 1 is next to the 3 and the 5.
- The 2 is to the left of the 4.

For example, one of the numbers that Parker writes down is 25134. Compute the sum of the smallest number and the greatest number that Parker writes down.

Answer: 74478

Solution: Since the 1 is next to the 3 and 5, there must be a continuous block of 315 or 513. After placing this block in the first three digits, the middle three digits, or the last three digits, there is only one way to place the 2 and the 4, since both 2 and 4 must fill in the remaining digits and 2 must be to the left of 4. Therefore, the only possible numbers are:

$$31524, 51324, 23154, 25134, 24315, 24513.$$

The sum of the least number and the greatest number is $23154 + 51324 = \boxed{74478}$.

4. Carolina has a jar of red, green, and white jelly beans. She replaces 400 red beans with green ones, then 200 green beans with white ones. Initially, the percentage of red beans minus the percentage of green beans was $X\%$. After the replacements, this difference became $(X - 2.5)\%$. Compute the number of beans in the jar.

   Answer: $24,000$

   Solution: Overall, the number of red jelly beans decreased by 400. The number of green jelly beans first increased by 400 after the first operation and then decreased by 200 by the second operation, leaving an overall 200 increase. Therefore, the overall change in the difference of red and green jelly beans is 600 jelly beans. Since this is 2.5% of the entire jar, there are

$$\frac{100}{2.5} \cdot 600 = \boxed{24,000}$$

   total jelly beans in the jar.

   REMARK: One can let r, g, and w be the number of red, green, and white jelly beans respectively. Then after the first operation we have $r - 400$, $g + 400$, and w red, green, and white jelly beans respectively. After the second operation, we have $r - 400$, $g + 200$, and $w + 200$ red, green, and white jelly beans respectively. Then since $(r - 400) - (g + 200) = -600$, this uses algebra to formalize our argument above.

5. The boxes in the expression below are filled with the numbers 3, 4, 5, 6, and 7, so that each number is used exactly once, and the value of the expression is a prime number. Compute the value of the expression.

$$\square \times \square \times \square + \square \times \square$$

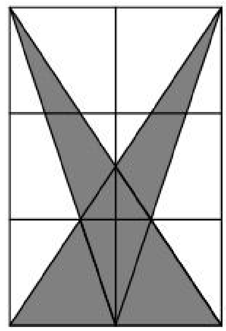
   Answer: 107

   Solution: The expression is a sum of two products. If the two products were to share any common factors, then the expression would be divisible by that common factor, which would make it not prime. Thus, 6 must be in the same product as 3, and also in the same product as 4. Therefore, the expression must be

$$3 \times 4 \times 6 + 5 \times 7 = 72 + 35 = \boxed{107}.$$

   Indeed, 107 is a prime number.

6. A 3×2 rectangle is divided into grid lines spaced one unit apart. The vertices on the grid are connected to form two overlapping triangles, both of which are shaded. Compute the area of the shaded region.



Answer: $\frac{5}{2}$

Solution: We subtract the white areas to determine the area of the shaded region. Consider the two congruent white triangles: each of them can be computed by subtracting the area of the smaller shaded triangle with length 1 and height 1 from the triangle with length 1 and height 3. The third white triangle has a base of length 2 and a height of length $\frac{3}{2}$ by symmetry. Subtracting the total area from the areas of the white triangles, we obtain the area of the shaded region:

$$2 \cdot 3 - \left(2 \cdot \frac{1 \cdot 3}{2} - \frac{1 \cdot 1}{2}\right) - \frac{2 \cdot \frac{3}{2}}{2} = \boxed{\frac{5}{2}}$$

7. Six friends—Andrea, Blake, Camila, Dean, Ethan, and Francis—drink a combined total of 32 cups of boba. Each person drinks at least one cup, no two friends drink the same number of cups, and each cup is fully finished by a single person. One friend, known as the "Boba Champion", drinks as many cups as all the others combined. Compute the product of the number of cups of boba consumed by the five friends who aren't the Boba Champion.

Answer: 144

Solution: Let a, b, c, d, e, f be the number of cups of boba each of the six friends drank. Without loss of generality, assume that f corresponds to the number of cups the Boba Champion drank. Then $a + b + c + d + e = f$ and $a + b + c + d + e + f = 2f = 32$ so $f = 16 = a + b + c + d + e$. Without loss of generality, assume that $a < b < c < d < e$. Since a, b, c, d, e are distinct positive integers, we get that their minimal sum is $1 + 2 + 3 + 4 + 5 = 15$. This is one less than our desired 16. So we need to increase one of the numbers by 1, while keeping the numbers distinct. The only possibility is to increase e to 6. Therefore, the only possible value for (a, b, c, d, e) is $(1, 2, 3, 4, 6)$. The answer is thus $1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 = \boxed{144}$.

8. Cindy goes to sleep at midnight (12:00 A.M.) and wakes up after n hours, where n is an integer between 1 and 24, inclusive. Her sister, Sleepy Windy, goes to sleep at the same time and wakes up n^2 hours later. It turns out that their shared analog clock looks the same to both sisters when each of them wakes up. Compute the sum of all possible values of n. (An analog clock displays all times between 12:00 and 11:59 with each cycle repeated twice a day, but the sisters cannot tell apart the A.M. and P.M. by looking at the analog clock).

Answer: 100

Solution: In order for Cindy and her sister to see the same clock when they wake up, the clock must look the same for n and n^2 after midnight. This means that we want to solve for $n \equiv n^2 \bmod 12$. Testing out all numbers from $1 - 12$, we see that $1, 4, 9$ and 12 all work. It follows that $13, 16, 21,$ and 24 also work, since $13 \equiv 1 \pmod{12}$, $16 \equiv 4 \pmod{12}$, $21 \equiv 9 \pmod{12}$, and $24 \equiv 12 \pmod{12}$. Therefore, the sum is $1 + 4 + 9 + 12 + 13 + 16 + 21 + 24 = \boxed{100}$.

9. Professor Matrix Tessellation Laplace writes the following equation on a blank blackboard:

$$z = 0.$$

One by one, each of his 2025 students edits the equation by adding a new term:

$$x_1 y_1 + z = 0$$

$$x_2 y_2 + x_1 y_1 + z = 0$$

$$\vdots$$

When all 2025 students are done, compute the number of characters written on the blackboard. These include digits, variable names, operations, and the equal sign (there is always only one equation on the board). For example, $x_{12} y_{12} + z = 0$ has 10 characters.

Answer: 20064

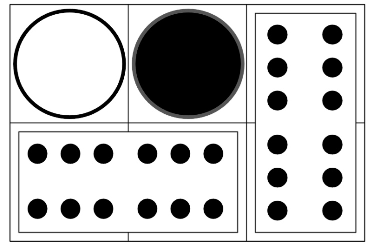
Solution: First, we know that x and y appear with at least one digit each 2025 times, giving $2025 \cdot 4 = 8100$ total symbols thus far. There are 90 two-digit numbers with one further symbol, 900 three-digit numbers with two further symbols, and $2025 - 1000 + 1 = 1026$ four-digit numbers with three further symbols. This gives a total of

$$(90 + 900 \cdot 2 + 1026 \cdot 3) \times 2 = 9936$$

further digit symbols. There are 2025 plus signs (one before the term with z all the way up to x_{2024}) and three symbols with $z = 0$. Thus, the final answer is

$$8100 + 9936 + 2025 + 3 = \boxed{20064}.$$

10. Kevin has two indistinguishable 2×1 dominoes (labeled $6 - 6$), and Ryan has two 1×1 distinguishable checkers (one black and one white). They take turns placing one piece at a time on a fixed 2×3 grid, aligning with the gridlines without overlap. Each piece must be placed face up, preserving its markings. Compute the number of possible final configurations. One example is shown (rotated or reflected boards are considered distinct).

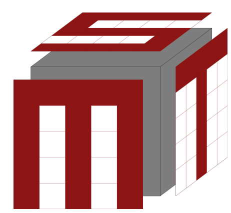


Answer: 22

Solution: We do casework on the number of vertical dominos:
Case One (2 vertical dominos). In this case, there are $\binom{3}{2} = 3$ ways to choose the positions of the vertical dominos. Afterwards, there are 2 ways to determine where the black checker goes. There are $3 \times 2 = 6$ boards in this case.
Case Two (1 vertical domino). In this case, there are 2 ways to choose the position of the vertical domino (one of the columns at the side but not the middle). Afterwards, there are 2 ways to determine the horizontal domino (top or bottom) and 2 ways to determine where the black checker goes. There are $2 \times 2 \times 2 = 8$ boards in this case.
Case Three (2 horizontal dominos). In this case, there are 2 ways to choose the position of the top horizontal domino and two for the bottom. Afterwards, there are 2 ways to determine where the black checker goes. There are $2 \times 2 \times 2 = 8$ boards in this case.

Adding up all three cases, the answer is $6 + 8 + 8 = \boxed{22}$.

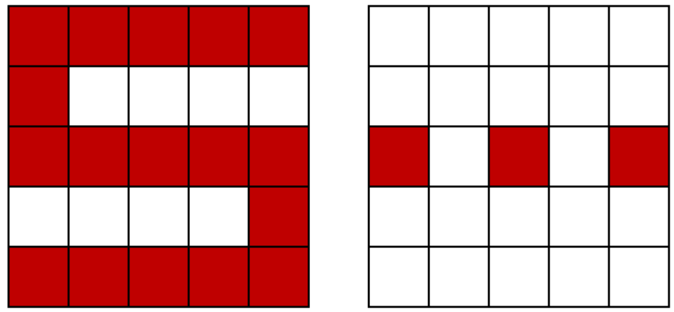
11. We have a $5 \times 5 \times 5$ cube made up of $1 \times 1 \times 1$ cardinal red and transparent pieces. When looking at the cube, we always see a cardinal red piece if it is behind any transparent pieces. When viewing each face straight-on, we can see the letters S, M, T on the top, front, and right side, respectively (as arranged in the diagram). Compute the maximum possible number of cardinal red pieces within this cube.
Note: cardinal red pieces are rendered black on your printed copy.



Answer: 29

Solution: We first determine which pieces must be transparent. If a square on the top/front/side is transparent, then all 5 pieces in the 5 layers must be transparent.

To maximize the number of cardinal red pieces, we count all pieces that are not definitely transparent as cardinal red pieces. The top layer has at most 17 cardinal red pieces, and each of the next 4 layers has 3 cardinal red pieces, so there are at most $17 + 4 \times 3 = \boxed{29}$ cardinal red pieces in total.



layer 1 layers 2 ~ 5

12. Compute the number of subsets A of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ which have the property that more than half of the elements in A are prime. For example, $A = \{2, 3, 4\}$ satisfies the condition. (A prime number is a positive integer greater than 1, whose only positive divisors are 1 and itself.)

Answer: 93

Solution: Note that exactly half of the elements are prime: $\{2, 3, 5, 7\}$. By symmetry, the number of subsets with more composite elements than prime elements is equal to the number of subsets with more primes than composites. We must subtract the cases where we consider subsets that have an equal number of composite and prime elements before dividing this number by two to account for the symmetry. For each of these cases we subtract, we must choose i elements from $\{2, 3, 5, 7\}$ and i elements from $\{1, 4, 6, 8\}$. Therefore, the answer is:

$$\frac{2^8 - \sum_{i=0}^{4} \binom{4}{i}^2}{2} = \boxed{93}.$$

13. Compute the number of triples of integers $-8 < a, b, c < 8$ that satisfy the following statements:

$$ab < 0,$$
$$ac < 0,$$
$$bc + a < 0.$$

Answer: 41

Solution: From the first condition, we see that a and b have to be of opposite signs. Similarly, from the second condition, we see that a and c have to be of opposite signs. From this, we have two main cases to consider based on the sign of a. Note that $a \neq 0$ since then $ab = 0$.

Either case 1 holds: $a > 0, b < 0, c < 0$, or case 2 holds: $a < 0, b > 0, c > 0$.

Case 1: Because $b, c < 0$, then $bc > 0$. Then, $a > 0$, so $bc + a > 0$. This contradicts the third condition, so no triples come from here.

Case 2: We rewrite the third condition as $bc < -a$. We now count the triples based off of the seven possible value of a (from −1 to −7).

$a = -1$ or $-a = 1$: $bc < 1$. No pairs (b, c) work because at minimum, $bc = 1 \cdot 1 = 1$.

$a = -2$ or $-a = 2$: $bc < 2$. There is 1 possible value of bc corresponding to 1 possible pair: $(b, c) = (1, 1)$.

$a = -3$ or $-a = 3$: $bc < 3$. There are 2 possible values of bc corresponding to 3 possible pairs: $(b, c) = (1, 1), (1, 2), (2, 1)$.

$a = -4$ or $-a = 4$: $bc < 4$. There are 3 possible values of bc corresponding to 5 possible pairs: $(b, c) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1)$.

$a = -5$ or $-a = 5$: $bc < 5$. There are 4 possible values of bc corresponding to 8 possible pairs: $(b, c) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 2)$.

$a = -6$ or $-a = 6$: There are 5 possible values of bc corresponding to 10 possible pairs: $(b, c) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 2), (1, 5), (5, 1)$.

$a = -7$ or $-a = 7$: There are 6 possible values of bc corresponding to 14 possible pairs: $(b, c) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 2), (1, 5), (5, 1), (1, 6), (6, 1), (2, 3), (3, 2)$.

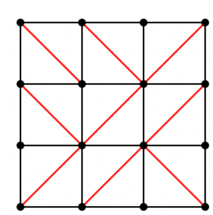
Adding all the cases we see that there are $1 + 3 + 5 + 8 + 10 + 14 = \boxed{41}$ possible triples.

14. Professor Matrix Tessellation Laplace rides his bike to his classroom everyday. To stop the Grand Theft Pedal in Stanford, he locks his bike using a 4-digit lock, and then shuffles his lock to a random combination. Each digit on the lock is a digit from 0 to 9 inclusive, and a flick is defined as changing a single digit by 1 (or between 0 and 9). Given that Professor Matrix Tessellation Laplace always finds the shortest combination of flicks, compute the average number of flicks he needs to unlock his bike.

    Answer: 10

    Solution: Since each digit is the same and independent, the number of flicks he needs to unlock his bike is 4 times the number of flicks to match a single digit. Furthermore, without loss of generality, we may assume that Professor Matrix Tessellation Laplace's password is 0000. If the digit after shuffled is $d > 5$, then the number of flicks needed is $10 - d$; otherwise, d flicks is needed. As such, the expected number of flicks for one digit is $\frac{0+1+2+3+4+5+4+3+2+1}{10} = 2.5$. Thus, the expected number of flicks for unlocking is $4 \cdot 2.5 = \boxed{10}$.

15. In a 3×3 grid, exactly one of the two possible diagonals is drawn in each of the nine cells. Compute the number of ways to choose the diagonals such that there is a continuous path along diagonals from the bottom-left vertex to the top-right vertex. One such grid is shown.



    Answer: 79

    Solution: First, observe that the bottom-left and top-right cells must be fixed and have their diagonals oriented to the right; otherwise, it would be impossible to start or end the path. Now we perform casework on the middle diagonal. If the middle diagonal faces right, then the path is automatically connected and we have 2^6 possibilities for the remaining cells (their choices are independent, as the path is already determined). If the middle diagonal is oriented to the left, then to obtain a connected path, the diagonals in the remaining cells must form one of two specific patterns: either a path that goes up and right, up and left, up and right, down and right, up and right, or alternatively, a path that goes up and right, down and right, up and right, up and left, up and right. There are 2^3 possibilities for the remaining cells after choosing one of these two paths, but we subtract 1 since we overcounted the possibility where both paths are created. In conclusion, there are $2^6 + 2^3 + 2^3 - 1 = \boxed{79}$ possible paths.

16. Let Ω be a circle. Draw two radii r_1 and r_2 of Ω that form a 40° angle, and let ω be a circle which is tangent to both of these radii and internally tangent to Ω. Let ω be tangent to Ω and r_1 at A and B, respectively, and let r_1 meet Ω at C. Compute $\angle ABC$.

    Answer: 55°

    Solution: Let O_Ω and O_ω be the centers of Ω and ω, respectively, and let $\theta = 40^\circ$ be the sector angle. Furthermore, we have that O_Ω, O_ω, and A are collinear. Note that by angle chasing we have

$$\begin{aligned} \angle O_\omega O_\Omega B &= 90 - \tfrac{\theta}{2} && \text{(as } \omega \text{ is tangent to } r_1) \\ \angle AO_\omega B &= 90 + \tfrac{\theta}{2} && \text{(by collinearity of } O_\Omega, O_\omega, \text{and } A) \\ \angle O_\omega BA &= 45 - \tfrac{\theta}{4} && \text{(since } O_\omega AB \text{ is isosceles)} \\ \angle ABC &= 45 + \tfrac{\theta}{4} && \text{(as } \omega \text{ is tangent to } r_1) \end{aligned}$$

    so we have $\angle ABC = 45 + \frac{40}{4} = \boxed{55^\circ}$.

17. Compute the sum of the real roots of $(x^2 + 16x + 48)(x^2 + 24x + 128) = 44^{2^{2025}}$. (Roots are counted with multiplicity.)

    Answer: -20

    Solution: Factor the quadratic expressions and observe that we have $(x + 4)(x + 12)(x + 8)(x + 16) = 44^{2^{2025}}$. Expand the factorized pairs $(x + 4)(x + 16) = x^2 + 20x + 64$ and $(x + 8)(x + 12) = x^2 + 20x + 96$. Noticing the symmetry, substitute $y = x^2 + 20x + 80$. Then, we have

$$(y - 16)(y + 16) = 44^{2^{2025}} \Rightarrow y^2 - 256 = 44^{2^{2025}} \Rightarrow y^2 = 256 + 44^{2^{2025}}.$$

    We define $c = \sqrt{256 + 44^{2^{2025}}}$, so that $y = \pm c$. For our purposes, it suffices to note that c is a very large number, which we can formally express as $c > 1000$. Now, we have two ways to finish:

    SOLUTION ONE. Solving for x, we obtain either $x^2 + 20x + 80 - c = 0$ or $x^2 + 20x + 80 + c = 0$. Since $20^2 - 4(80 + c) < 400 - 4(1000) < 0$, applying the quadratic formula shows that $x^2 + 20x + 80 + c = 0$ has no real solutions. On the other hand, by the same quadratic formula argument, since $400 - 4(80 - c) = 80 + 4c > 4000 > 0$, it follows that $x^2 + 20x + 80 - c = 0$ does have real solutions (this method is known as checking the discriminant of a quadratic). By Vieta's formulas on this quadratic, the sum of the real roots is $\boxed{-20}$.

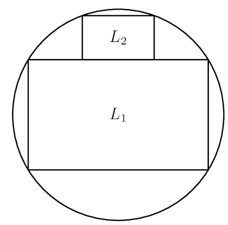
    SOLUTION TWO. Instead of applying the quadratic formula, we may complete the square and observe that $x^2 + 20x + 80 = (x + 10)^2 - 20 \geq -20$ for real x. Since c is a large number, $x^2 + 20x + 80$ cannot be $-c$. However, since $c > -20$, it follows that $x^2 + 20x + 80 = c$ has two real roots. This provides an alternative way to determine that the real roots satisfy $x^2 + 20x + 80 = c$, from which Vieta's formulas gives an answer of $\boxed{-20}$.

SOLUTION THREE. This is a separate solution from the first two. Recall that the equation can be rewritten as $(x+4)(x+12)(x+8)(x+16) = 44^{2^{2025}}$. Now, let $y = x + 10$. Rewriting the equation in terms of y, we obtain $(y-6)(y+2)(y-2)(y+6) = (y-2)(y+2)(y-6)(y+6) = (y^2-4)(y^2-36) = 44^{2^{2025}}$. Let $z = y^2$ so that we have

$$(z-4)(z-36) - 44^{2^{2025}} = z^2 - 40z + 144 - 44^{2^{2025}} = 0.$$

By the quadratic formula, $z = \frac{40 \pm \sqrt{(-40)^2 + 4(44^{2^{2025}} - 144)}}{2}$. Since $z = y^2 \geq 0$ whenever x, and hence y, is real, we only consider the positive value of z. For this positive value, the two possible values y are of form y_1, y_2 with $y_1 = -y_2$, since they square to the same value. Then if x_1, x_2 are the real roots corresponding to y_1, y_2 respectively, $(x_1 + x_2) + 20 = (x_1 + 10) + (x_2 + 10) = y_1 + y_2 = 0$ and $x_1 + x_2 = \boxed{-20}$.

18. In the diagram shown, the rectangles L_1 and L_2 are similar, where corresponding sides of L_1 are $\frac{5}{2}$ times the length of the corresponding sides of L_2. Rectangle L_1 is inscribed in a circle ω, and rectangle L_2 meets ω at two points, with its long side lying on the long side of L_1. If r and s are the side lengths of L_1, where $s > r$, compute $\frac{s}{r}$.



Answer: $\frac{2\sqrt{6}}{3}$

Solution: Make a copy of L_2 by rotating it 180 degrees about the center of ω. Connect the two copies via vertical segments to make one tall rectangle. This new rectangle has side lengths $(2 \cdot \frac{2}{5} + 1)r$ and $\frac{2}{5}s$, and its diagonal has the same length as L_1's (which is the diameter of ω). We use the Pythagorean Theorem to find both values: $r^2 + s^2$ and $((2 \cdot \frac{2}{5} + 1)r)^2 + (\frac{2}{5}s)^2$. Setting these equal and solving gives that $r = \sqrt{\frac{3}{8}}s$, so the ratio is $\sqrt{8/3} = \boxed{\frac{2\sqrt{6}}{3}}$.

19. An integer $m > 1$ is called *elementary* if its first digit is 1, its last digit is 1, and every other digit is 0. Let S be the set of the 2025 smallest elementary integers. Compute the greatest possible size of a subset T of S such that any two distinct elements in T are pairwise coprime.

Answer: 11

Solution: Let $s_i = 10^i + 1$ for positive integers i. Then, S is the set $\{s_1, s_2, ...s_{2025}\}$. Note that $10^{2n+1}x + 1$ is divisible by $10^x + 1$ for all positive integer n, since $(10^{2n+1}x + 1) - (10^x + 1) = 10^{2n+1}x - 10^x = 10^x(10^{2nx} - 1) = 10^x(10^x + 1)(10^{(2n-1)x} - 10^{(2n-2)x} + ... + x - 1)$. This means in T we cannot have two elements s_j and s_k such that j and k are odd multiples of the same integer. Then, we can only have one element $s_i \in S$ such that $s_i \equiv 2^k \pmod{2^{k+1}}$ for all integers $k \geq 0$. Since $2^{10} < 2025 < 2^{11}$, this means that the largest possible size of such a subset T is $\boxed{11}$. One such subset T is $\{10^{2^0} + 1, 10^{2^1} + 1, ..., 10^{2^{10}} + 1\}$.

20. Given real numbers s, m, and t such that $s^2 + m^2 + t^2 = 8$, $s^3 + m^3 + t^3 = 11$, and $s^4 + m^4 + t^4 = 25$, compute the value of $(s+m+t)(s+m-t)(s-m+t)(s-m-t)$.

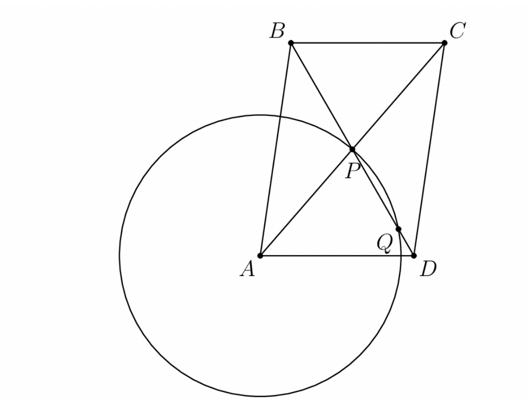
Answer: -14

Solution: We use the identity $s^4+m^4+t^4=(s^2+m^2+t^2)^2-2(s^2m^2+s^2t^2+m^2t^2)$ to express $s^2m^2+s^2t^2+m^2t^2$ as $\frac{(s^2+m^2+t^2)^2}{2}-\frac{s^4+m^4+t^4}{2}$. Therefore:

$$\begin{aligned}
&(s+m+t)(s+m-t)(s-m+t)(s-m-t)\\
&=s^4+m^4+t^4-2(s^2m^2+s^2t^2+m^2t^2)\\
&=s^4+m^4+t^4-2\left[\frac{(s^2+m^2+t^2)^2}{2}-\frac{s^4+m^4+t^4}{2}\right]\\
&=2(s^4+m^4+t^4)-(s^2+m^2+t^2)^2\\
&=2(25)-8^2\\
&=50-64\\
&=\boxed{-14}.
\end{aligned}$$

REMARK. Alternatively, one can directly substitute $2(s^2m^2+s^2t^2+m^2t^2)$ as $(s^2+m^2+t^2)^2-(s^4+m^4+t^4)$ above without first solving for $s^2m^2+s^2t^2+m^2t^2$.

21. Let $ABCD$ be a parallelogram with $AB>BC$ and diagonals intersecting at point P. Circle ω has its center at point A, radius AP, and intersects BD again at point Q on line segment PD. If $PQ=3$, $QD=1$, and $ABCD$ has integer side lengths, compute the length of AC.



Answer: $2\sqrt{21}$

Solution: Let ω have radius r, meet AD at M, and meet AB at N. Then it follows that

$$\text{Pow}_\omega(D)=MD(MD+2r)=1(1+3)=4$$

and setting this equation as a quadratic in MD, we have $MD=\sqrt{r^2+4}-r\implies AD=\sqrt{r^2+4}$. Similarly, we find that $AB=\sqrt{r^2+28}$. We want AB and AD to both be positive integers; note that $AB^2-AD^2=24$ is satisfied by only two pairs of positive integers: $(7,5)$ and $(5,1)$. The latter pair violates the triangle inequality as $AB+AD=6<2\cdot 4=8$. Thus $\{AD,AB\}=\{5,7\}$, and $AC=2r=2\sqrt{5^2-4}=\boxed{2\sqrt{21}}$.

22. Consider an axis-aligned 4 dimensional cube with side length 4, subdivided into 4^4 unit cubes. Cubey the penguin starts at a random unit cube. Each second, he can move to an adjacent unit cube along any one of the four axes. A fish is independently placed in a uniformly random unit cube (possibly the same one as Cubey's starting position) and the fish does not move (it's a fish out of water). Compute the expected time in seconds that Cubey will take to catch the fish, assuming he moves optimally to reach the fish as fast as possible.

Answer: 5

Solution: Let Cubey's location be (x_1, y_1, z_1, w_1) and the fish's location be (x_2, y_2, z_2, w_2). It will take $d = |\ x_1 - x_2| + |\ y_1 - y_2| + |\ z_1 - z_2| + |\ w_1 - w_2|$ seconds for him to reach the fish. Then, we can use linearity of expectation and symmetry to write the expected distance as

$$\mathbb{E}[d] = \mathbb{E}[|\ x_1 - x_2|] + \mathbb{E}[|\ y_1 - y_2|] + \mathbb{E}[|\ z_1 - z_2|] + \mathbb{E}[|\ w_1 - w_2|] = 4 \cdot \mathbb{E}(|\ x_1 - x_2|)$$

If x_1 and x_2 can be any random integer from 1 to 4, then

$$\mathbb{E}[|\ x_1 - x_2|] = \frac{1}{16} \sum_{i_1=1}^{4} \sum_{i_2=1}^{4} |\ i_1 - i_2| = \frac{2}{16} \sum_{i_1=2}^{4} \sum_{i_2=1}^{i_1-1} (i_1 - i_2)$$

$$= \frac{1}{8} \sum_{i_1=2}^{4} (i_1 - 1)i_1 - \frac{i_1(i_1 - 1)}{2} = \frac{1}{8} \sum_{i_1=2}^{4} \frac{i_1^2 - i_1}{2} = \frac{1}{8}(1 + 3 + 6) = \frac{5}{4}.$$

This gives a final answer of $4 \cdot \frac{5}{4} = \boxed{5}$. An easier way to compute this sum is to manually write out the 16 possible pairs of x_1, x_2.

23. Compute the number of integers between 1 and 2025 inclusive that *cannot* be represented as $x(\lceil x \rceil + \lfloor x \rfloor)$ for any positive real number x.
(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x and $\lceil x \rceil$ is the smallest integer not less than x.)

Answer: 1002

Solution: Let $f(x) = x(\lceil x \rceil + \lfloor x \rfloor)$. We will instead compute the number of integers that are expressible as $f(x)$ for some x.

Now, let us partition the space of all positive real numbers by their floor: in particular, suppose that $\lfloor x \rfloor = n$. If $x = n$, then $f(x) = 2n^2$ (as $\lceil n \rceil = n$).

Else, if $n < x < n + 1$, we claim that the integers which can be expressed as $f(x)$ are exactly the integers from $2n^2 + n + 1$ through $2n^2 + 3n$, inclusive. Indeed, in this interval we have that $f(x) = x(2n + 1)$. Evaluating at $x = n$ and $n + 1$ gives $2n^2 + n$ and $2n^2 + 3n + 1$, respectively. Since $f(x)$ is strictly increasing and continuous on this range, it follows that the expressible integers for $x \in (n, n + 1)$ (i.e., excluding the endpoints) are exactly $[2n^2 + n + 1, 2n^2 + 3n]$.

Therefore, we have two types of expressible integers: those of the above form and those of the form $2n^2$. Note that these two types of ranges never intersect, as

$$2n^2 + 3n < 2(n + 1)^2 = 2n^2 + 4n + 2 < 2(n + 1)^2 + (n + 1) + 1.$$

The largest n we have to consider is 31, as $2 \cdot 32^2 = 2048 > 2025$ and $2 \cdot 31^2 + 3 \cdot 31 = 2015 \leq 2025$. For each n, the total number of integers expressible by its ranges is $1 + ((2n^2 + 3n) - (2n^2 + n + 1) + 1) = 2n + 1$. Therefore, the total number of expressible integers is

$$\sum_{n=1}^{31} 2n + 1 = 31 + 31 \cdot 32 = 1023$$

and our final answer is $2025 - 1023 = \boxed{1002}$.

24. Professor Matrix Tessellation Laplace arranges all of the integers from 1 to 99999 (inclusive) in the order that maximizes the gigantic number formed by concatenating them. In his order, the number 2025 appears as the i-th integer from left to right. Find i.

(Formally, he sets the permutation $\sigma = (\sigma_1, ..., \sigma_{99999})$ of $\{1, 2, ..., 99999\}$ such that the value of the concatenation $\overline{\sigma_1 \sigma_2 ... \sigma_{99999}}$ is maximized. Find the index i such that $\sigma_i = 2025$.)

Answer: 88607

Solution: Denote the concatenation of two numbers a and b with $a \oplus b$. For any two numbers a and b, let the number of digits of a and b be m and n respectively. $a \oplus b = a(10)^n + b$, $b \oplus a = b(10)^m + a$. As such,

$$a \oplus b > b \oplus a \Longleftrightarrow a(10)^n + b > b(10)^m + a \Longleftrightarrow \frac{a}{10^m - 1} > \frac{b}{10^n - 1}$$

In other words, for any integer k with d digits, we are ranking k based on $\frac{k}{10^d - 1}$ from largest to smallest. Since 2025 has a unique result for this expression, we can determine its position in the arrangement by calculating the number of integers from 1 to 99999 with this expression computed to be larger than 2025/9999. By case work: [label=]

1. 1 digit: from 2 to 9 $\Longrightarrow$ 8 cases.
2. 2 digit: from 21 to 99 $\Longrightarrow$ 79 cases.
3. 3 digit: from 203 to 999 $\Longrightarrow$ 797 cases.
4. 4 digit: from 2026 to 9999 $\Longrightarrow$ 7974 cases.
5. 5 digit: from 20252 to 99999 $\Longrightarrow$ 79748 cases.

There are $8 + 79 + 797 + 7974 + 79748 = 88606$ cases in total, so 2025 is in the $\boxed{88607}$-th position.

25. Let $ABCD$ be a square centered at E with side length 10. Point P lies on the extension of line CD past point D. There exists a point Q on line AP such that lines EQ, BP, and AD are concurrent. If $EQ = 7\sqrt{2}$, compute $|AQ - DQ|$.

Answer: 2

Solution: SOLUTION 1: Extend EQ to meet AB at N and CD at M, and let T be the intersection of AD, BP, and MN.

Since BN and MP are parallel,

$$AN : CM = AE : CE = 1 : 1,$$
$$AN : DM = AT : DT = AB : DP,$$
$$AN : MP = AQ : QP$$

by similar triangles.

Let DP = x, so $AN : DM = 10 : x$.

Because $DM = CD + CM = AN + 10$, $AN = \frac{100}{x-10}$ and $DM = \frac{10x}{x-10}$.

Because $MP = DM + DP = \frac{x^2}{x-10}$, $AQ : QP = AN : MP = 100 : x^2$.

Since the right triangle ADP has side length ratio $AD : DP = 10 : x$, $\angle AQP = 90^\circ$, so $AEDQ$ is a cyclic quadrilateral.

Since $AE = DE = 5\sqrt{2}$ and $EQ = 7\sqrt{2}$, either $AQ = 6, DQ = 8$ or $AQ = 8, DQ = 6$, so $\mid AQ - DQ \mid = \boxed{2}$.

SOLUTION 2: Let us impose a coordinate system on this configuration with $A = (10, 10)$, $B = (0, 10)$, $C = (0, 0)$, $D = (10, 0)$, and $E = (5, 5)$. Let $P = (10 + p, 0)$ for some $p > 0$. Since Q lies on AP, $Q = (10 + tp, 10 - 10t)$ for some $0 < t < 1$. Let X be the intersection of BP and AD. Since lines BP and AD have equations $y = 10 - \frac{10x}{p+10}$ and $x = 10$. Solving the equations gives us:

$$X = \left(10, \frac{10p}{p + 10}\right).$$

The line EQ and AD have equations $y = \left(\frac{5-10t}{5+tp}\right)(x - 5) + 5$ and $x = 10$, so it intersects at coordinate point $\left(10, \frac{50-50t+5tp}{5+tp}\right)$. By the concurrency condition, $X = \left(10, \frac{10p}{p+10}\right) = \left(10, \frac{50-50t+5tp}{5+tp}\right)$. Solving for t

$$\frac{10p}{p + 10} = \frac{50 - 50t + 5tp}{5 + tp}$$
$$(10p)(5 + tp) = (50 - 50t + 5tp)(p + 10)$$
$$10p^2t + 50p = 5p^2t + 50p - 500t + 500$$
$$(5p^2 + 500)t = 500$$
$$t = \frac{500}{5p^2 + 500} = \frac{100}{p^2 + 100}$$

Finally, we can solve for p using $EQ = 7\sqrt{2}$. Since $E = (5, 5)$ and $Q = (10 + tp, 10 - 10t)$

$$
\begin{aligned}
\|EQ\| = \sqrt{(5+tp)^2+(5-10t)^2} &= 7\sqrt{2}\\
25+t^2p^2+10tp+25-100t+100t^2 &= 98\\
t^2(p^2+100)+10tp-100t &= 48\\
100t+10tp-100t &= 48\\
10tp = \left(\frac{1000}{p^2+100}\right)p &= 48\\
1000p &= 48(p^2+100)\\
48p^2-1000p+4800 &= 0\\
6p^2-125p+600 = (3p-40)(2p-15) &= 0\\
p = \frac{40}{3} \quad \text{or} \quad p &= \frac{15}{2}.
\end{aligned}
$$

The distances from $Q = (10+tp, 10-10t)$ to $A = (10, 10)$ and $D = (10, 0)$ are

$$
\begin{aligned}
\|AQ\| &= \sqrt{t^2p^2+100t^2}\\
&= t\sqrt{p^2+100}\\
&= \frac{100}{\sqrt{p^2+100}}
\end{aligned}
$$

and

$$
\begin{aligned}
\|DQ\| &= \sqrt{t^2p^2+(10-10t)^2}\\
&= \sqrt{t^2p^2+100+100t^2-200t}\\
&= \sqrt{t^2(p^2+100)+100-200t}\\
&= \sqrt{100t+100-200t}\\
&= 10\sqrt{1-t}\\
&= 10\sqrt{\frac{p^2}{p^2+100}}\\
&= \frac{10p}{\sqrt{p^2+100}}.
\end{aligned}
$$

Hence,

$$|\| AQ \| - \| DQ \|| = \frac{10 \mid 10-p \mid}{\sqrt{p^2+100}}.$$

Plugging $p = \frac{40}{3}$ yields

$$\mid 10-p \mid = \frac{10}{3} \quad \text{and} \quad \sqrt{p^2+100} = \frac{50}{3},$$

so that

$$|\| AQ \| - \| DQ \|| = \frac{10 \cdot \frac{10}{3}}{\frac{50}{3}} = \frac{100}{50} = \boxed{2}.$$

Similarly, one can plug in $p = \frac{15}{2}$ and get the same answer.

TB. *This is an estimation question used for tiebreaking purposes. Ties on this test will be broken by absolute distance from the correct answer to this question.*

Let s_0 be a square. One diagonal of s_0 is chosen, and a square s_1 is drawn with this diagonal as one of its sides. There are four possible such squares s_1; one is chosen out of these four uniformly at random. Choose s_2 in the same manner (so that it has a side which is a diagonal of s_1), and continue similarly with $s_3, s_4, \ldots$. Let p be the probability that square s_{20} shares an interior with s_0 (in other words, the area of their intersection is positive).
Estimate p in the decimal form $0.abcdefgh$, where a, b, c, d, e, f, g, h are decimal digits each between 0 and 9, inclusive.

Answer: 0.41666698

Solution: