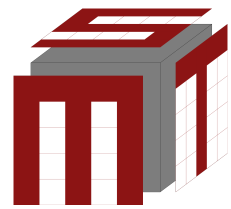
1. Let $\triangle ABC$ be an equilateral triangle with side length 6. Points D, E, and F lie on sides AB, BC, and CA respectively such that $\triangle DEF$ is equilateral and has area equal to $\triangle FAD$. Compute the side length of $\triangle DEF$.

   Answer: 3

   Solution: By symmetry, the triangles $\triangle FAD$, $\triangle DBE$, and $\triangle ECF$ are congruent and therefore have the same area. Since $\triangle DEF$ has area equal to each of these triangles, and they partition $\triangle ABC$, it follows that the area of $\triangle DEF$ is $\frac{1}{4}$ of that of $\triangle ABC$. That is, $\triangle DEF$ is similar to $\triangle ABC$ with a $\sqrt{\frac{1}{4}} = \frac{1}{2}$ ratio, and the side length of $\triangle DEF$ is $\frac{6}{2} = \boxed{3}$.

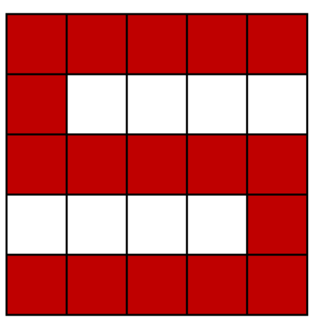
2. We have a $5 \times 5 \times 5$ cube made up of $1 \times 1 \times 1$ cardinal red and transparent pieces. When looking at the cube, we always see a cardinal red piece if it is behind any transparent pieces. When viewing each face straight-on, we can see the letters S, M, T on the top, front, and right side, respectively (as arranged in the diagram). Compute the maximum possible number of cardinal red pieces within this cube.
   Note: cardinal red pieces are rendered black on your printed copy.



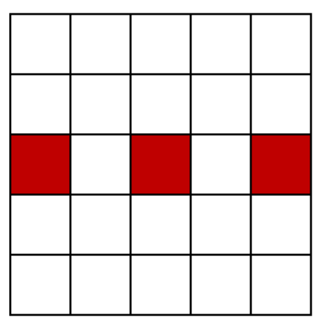
   Answer: 29

   Solution: We first determine which pieces must be transparent. If a square on the top/front/side is transparent, then all 5 pieces in the 5 layers must be transparent.

   To maximize the number of cardinal red pieces, we count all pieces that are not definitely transparent as cardinal red pieces. The top layer has at most 17 cardinal red pieces, and each of the next 4 layers has 3 cardinal red pieces, so there are at most $17 + 4 \times 3 = \boxed{29}$ cardinal red pieces in total.



layer 1



layers 2 ~ 5

3. Let Ω be a circle. Draw two radii r_1 and r_2 of Ω that form a 40° angle, and let ω be a circle which is tangent to both of these radii and internally tangent to Ω. Let ω be tangent to Ω and r_1 at A and B, respectively, and let r_1 meet Ω at C. Compute $\angle ABC$.

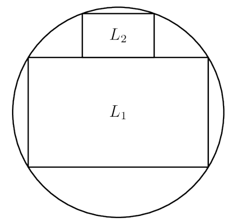
   Answer: 55°

   Solution: Let O_Ω and O_ω be the centers of Ω and ω, respectively, and let $\theta = 40^\circ$ be the sector angle. Furthermore, we have that O_Ω, O_ω, and A are collinear. Note that by angle chasing we have

$$\angle O_\omega O_\Omega B = 90 - \tfrac{\theta}{2} \qquad \text{(as } \omega \text{ is tangent to } r_1\text{)}$$
$$\angle AO_\omega B = 90 + \tfrac{\theta}{2} \text{(by collinearity of } O_\Omega, O_\omega, \text{and } A\text{)}$$
$$\angle O_\omega BA = 45 - \tfrac{\theta}{4} \qquad \text{(since } O_\omega AB \text{ is isosceles)}$$
$$\angle ABC = 45 + \tfrac{\theta}{4} \qquad \text{(as } \omega \text{ is tangent to } r_1\text{)}$$

so we have $\angle ABC = 45 + \frac{40}{4} = \boxed{55^\circ}$.

4. In the diagram shown, the rectangles L_1 and L_2 are similar, where corresponding sides of L_1 are $\frac{5}{2}$ times the length of the corresponding sides of L_2. Rectangle L_1 is inscribed in a circle ω, and rectangle L_2 meets ω at two points, with its long side lying on the long side of L_1. If r and s are the side lengths of L_1, where $s > r$, compute $\frac{s}{r}$.



Answer: $\frac{2\sqrt{6}}{3}$

Solution: Make a copy of L_2 by rotating it 180 degrees about the center of ω. Connect the two copies via vertical segments to make one tall rectangle. This new rectangle has side lengths $(2 \cdot \frac{2}{5} + 1)r$ and $\frac{2}{5}s$, and its diagonal has the same length as L_1's (which is the diameter of ω). We use the Pythagorean Theorem to find both values: $r^2 + s^2$ and $((2 \cdot \frac{2}{5} + 1)r)^2 + (\frac{2}{5}s)^2$. Setting these equal and solving gives that $r = \sqrt{\frac{3}{8}}s$, so the ratio is $\sqrt{8/3} = \boxed{\frac{2\sqrt{6}}{3}}$.

5. Let α and β be two angles in $[0, 2\pi)$, and let $A = (\cos\alpha, \sin\alpha)$, $B = (\cos\beta, \sin\beta)$ and $C = (\cos\alpha + \cos\beta, \sin\alpha + \sin\beta)$. If one of the points has coordinates $\left(\frac{32}{25}, \frac{24}{25}\right)$, compute the area of $\triangle ABC$.

Answer: $\frac{12}{25}$

Solution: Since $\cos^2\alpha + \sin^2\alpha = \cos^2\beta + \sin^2\beta = 1$, $OA = OB = 1$. Furthermore, $AC = \sqrt{((\cos\alpha + \cos\beta) - (\cos\alpha))^2 + ((\sin\alpha + \sin\beta) - (\sin\alpha))^2} = \sqrt{(\cos\beta)^2 + (\sin\beta)^2} = OB = 1$, and similarly $BC = \sqrt{((\cos\alpha + \cos\beta) - (\cos\beta))^2 + ((\sin\alpha + \sin\beta) - (\sin\beta))^2} = \sqrt{(\cos\alpha)^2 + (\sin\alpha)^2} = OA = 1$, so $OABC$ is a rhombus with side length 1 (this can also be proved by noting that C is the vector sum of A and B). Because $(\frac{32}{25}, \frac{24}{25})$ has distance from the origin $\sqrt{(\frac{32}{25})^2 + (\frac{24}{25})^2} = \frac{8}{5} > 1$, it must be point C. Since $OB = OC$, OBC is an isosceles triangle. As the diagonals of a rhombus perpendicularly bisect each other, $\frac{OC}{2} = \frac{4}{5}$ is the height from O to AB. Then $\triangle ABC = \triangle OAB = \frac{1}{2} \cdot OC \cdot \sqrt{1^2 - (\frac{OC}{2})^2} = \boxed{\frac{12}{25}}$.

6. Let $\mathcal{P}$ be a parabola with its vertex at A, and let B and C be two other distinct points on $\mathcal{P}$. The focus F of $\mathcal{P}$ is the centroid of $\triangle ABC$. If $FB = 3$, compute the area of $\triangle ABC$.

Answer: $\frac{54\sqrt{6}}{25}$

Solution: WLOG, the parabola as equation $y = ax^2$ for some positive, real a. We have $A = (0, 0)$ and $F = (0, \frac{1}{4a})$. Let $B = (b, ab^2)$ and $C = (c, ac^2)$. WLOG, $b > 0$, since we can reflect the diagram over $x = 0$. F is the centroid of $\triangle ABC$, so

$$\left(\frac{0+b+c}{3}, \frac{0+ab^2+ac^2}{3}\right) = \left(0, \frac{1}{4a}\right) \implies b = -c \implies b^2 = \frac{3}{8a^2}.$$

The directrix is $y = -\frac{1}{4a}$. By the definition of a parabola, the distance from B to the directrix is 3. This means

$$3 = ab^2 + \frac{1}{4a} = \frac{3}{8a} + \frac{1}{4a} = \frac{5}{8a} \implies a = \frac{5}{24}.$$

Now, we calculate

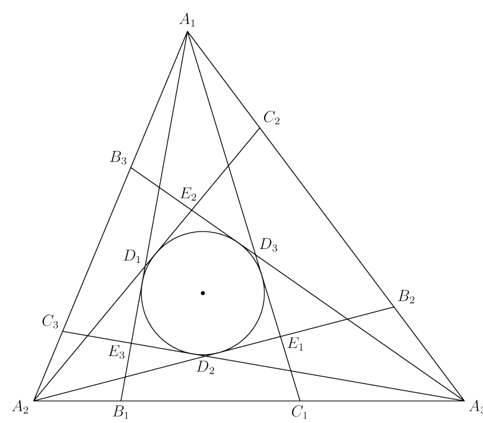
$$[ABC] = \frac{2b \cdot ab^2}{2} = ab^3 = \frac{a^3b^3}{a^2} = \frac{\left(\frac{\sqrt{3}}{\sqrt{8}}\right)^3}{\left(\frac{5}{24}\right)^2} = \frac{3\sqrt{3}\cdot 24^2}{16\sqrt{2}\cdot 25} = \boxed{\frac{54\sqrt{6}}{25}}.$$

SOLUTION #2 (Synthetic Finish): Let ℓ be the directrix of $\mathcal{P}$. Note that $FB = \text{dist}(B, \ell) = FC = \text{dist}(C, \ell) = 3$. It follows that BC and ℓ are parallel. Let M be the midpoint of BC and let $FM = t$. Then $FA = 2 \cdot FM = 2t$, and $\text{dist}(F, A) = \text{dist}(A, \ell) = 2t$. Thus

$$\text{dist}(\ell, M) = 5t = 3 = \text{dist}(B, \ell) \implies t = \frac{3}{5}$$

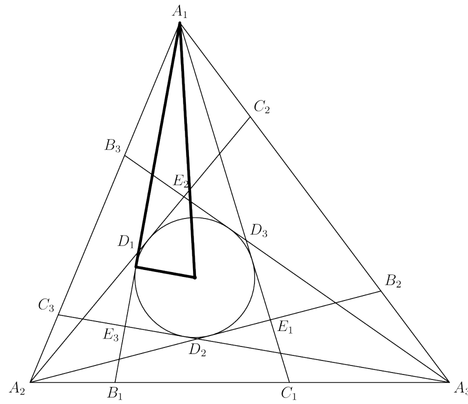
From here, we can compute $[ABC] = AM \cdot MB = \frac{9}{5}\sqrt{FB^2 - FM^2} = \frac{9}{5} \cdot \frac{6\sqrt{6}}{5} = \boxed{\frac{54\sqrt{6}}{25}}$.

7. Triangle $\triangle A_1A_2A_3$ has side lengths $A_1A_2 = 13$, $A_2A_3 = 14$, $A_3A_1 = 15$. Points B_i and C_i lie on $\overline{A_{i+1}A_{i+2}}$ such that $\angle B_iA_iC_i = 30^\circ$ for $1 \le i \le 3$, where $A_4 = A_1$ and $A_5 = A_2$. For $1 \le i \le 3$, let D_i be the intersection of $\overline{A_iB_i}$ and $\overline{A_{i+1}C_{i+1}}$ and E_i the intersection of $\overline{A_iC_i}$ and $\overline{A_{i+1}B_{i+1}}$, where $A_4 = A_1, B_4 = B_1, C_4 = C_1$. If $D_1E_3D_2E_1D_3E_2$ forms a convex hexagon with an inscribed circle of radius r, compute r.



Answer: $\frac{65\sqrt{6}-65\sqrt{2}}{32}$

Solution: Draw the radii from the center of the circle to the tangent points. Note that all 6 triangles as shown are congruent via AAS:

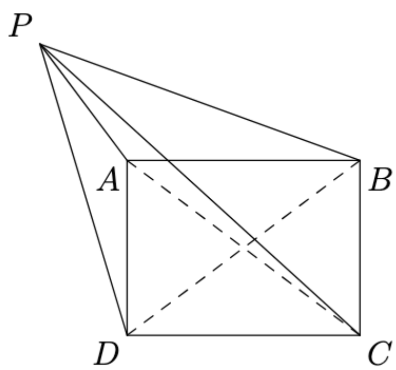


Hence, if R is the circumradius of $\triangle A_1A_2A_3$ and r is the radius of the small circle, then $r = R\sin(15^\circ) = \boxed{\frac{65\sqrt{6}-65\sqrt{2}}{32}}$.

8. Let $PABCD$ be a pyramid with rectangular base $ABCD$ and $\frac{\cos\angle APC}{\cos\angle BPD} = 2$. If $BP = 4$ and $DP = 9$, compute $AP + CP$.

Answer: $\sqrt{133}$

Solution:



Let P' be the projection of P onto the plane containing $ABCD$. By the British Flag Theorem,

$$(AP')^2 + (CP')^2 = (BP')^2 + (DP')^2.$$

It follows that

$$AP^2 + CP^2 = (AP')^2 + (PP')^2 + (CP')^2 + (PP')^2$$
$$= (BP')^2 + (PP')^2 + (DP')^2 + (PP')^2 = BP^2 + DP^2 = 4^2 + 9^2 = 97.$$

Note that $ABCD$ is a rectangle, so $AC = BD$. By the Law of Cosines on $\triangle APC$ and $\triangle BPD$,

$$AP^2 + CP^2 - 2\cdot AP\cdot CP\cdot\cos\angle APC = AC^2 = BD^2 = BP^2 + DP^2 - 2\cdot BP\cdot DP\cdot\cos\angle BPD$$

from which $AP\cdot CP = \frac{4\cdot 9}{2} = 18$. Now we have

$$AP + CP = \sqrt{AP^2 + 2 \cdot AP \cdot CP + CP^2} = \sqrt{97 + 36} = \boxed{\sqrt{133}}.$$

Note that an exact configuration works:

$$A = (0, 0, 0)$$
$$B = \left(\sqrt{\frac{\sqrt{8113} - 65}{2}}, 0, 0 \right)$$
$$C = \left(\sqrt{\frac{\sqrt{8113} - 65}{2}}, \sqrt{\frac{65 + \sqrt{8113}}{2}}, 0 \right)$$
$$D = \left(0, \sqrt{\frac{65 + \sqrt{8113}}{2}}, 0 \right)$$
$$P = \left(0, 0, \sqrt{\frac{97 - \sqrt{8113}}{2}} \right).$$

9. Let $ABCD$ be a square centered at E with side length 10. Point P lies on the extension of line CD past point D. There exists a point Q on line AP such that lines EQ, BP, and AD are concurrent. If $EQ = 7\sqrt{2}$, compute $|AQ - DQ|$.

Answer: 2

Solution: SOLUTION 1: Extend EQ to meet AB at N and CD at M, and let T be the intersection of AD, BP, and MN.

Since BN and MP are parallel,

$$AN : CM = AE : CE = 1 : 1,$$
$$AN : DM = AT : DT = AB : DP,$$
$$AN : MP = AQ : QP$$

by similar triangles.

Let DP = x, so $AN : DM = 10 : x$.

Because $DM = CD + CM = AN + 10$, $AN = \frac{100}{x-10}$ and $DM = \frac{10x}{x-10}$.

Because $MP = DM + DP = \frac{x^2}{x-10}$, $AQ : QP = AN : MP = 100 : x^2$.

Since the right triangle ADP has side length ratio $AD : DP = 10 : x$, $\angle AQP = 90^\circ$, so $AEDQ$ is a cyclic quadrilateral.

Since $AE = DE = 5\sqrt{2}$ and $EQ = 7\sqrt{2}$, either $AQ = 6, DQ = 8$ or $AQ = 8, DQ = 6$, so $\mid AQ - DQ \mid = \boxed{2}$.

SOLUTION 2: Let us impose a coordinate system on this configuration with $A = (10, 10)$, $B = (0, 10)$, $C = (0, 0)$, $D = (10, 0)$, and $E = (5, 5)$. Let $P = (10 + p, 0)$ for some $p > 0$. Since Q lies on AP, $Q = (10 + tp, 10 - 10t)$ for some $0 < t < 1$. Let X be the intersection of BP and AD. Since lines BP and AD have equations $y = 10 - \frac{10x}{p+10}$ and $x = 10$. Solving the equations gives us:

$$X = \left(10, \frac{10p}{p+10}\right).$$

The line EQ and AD have equations $y = \left(\frac{5-10t}{5+tp}\right)(x-5)+5$ and $x = 10$, so it intersects at coordinate point $\left(10, \frac{50-50t+5tp}{5+tp}\right)$. By the concurrency condition, $X = \left(10, \frac{10p}{p+10}\right) = \left(10, \frac{50-50t+5tp}{5+tp}\right)$. Solving for t

$$\begin{aligned}
\frac{10p}{p+10} &= \frac{50-50t+5tp}{5+tp} \\
(10p)(5+tp) &= (50-50t+5tp)(p+10) \\
10p^2t + 50p &= 5p^2t + 50p - 500t + 500 \\
(5p^2+500)t &= 500 \\
t &= \frac{500}{5p^2+500} = \frac{100}{p^2+100}
\end{aligned}$$

Finally, we can solve for p using $EQ = 7\sqrt{2}$. Since $E = (5,5)$ and $Q = (10+tp, 10-10t)$

$$\begin{aligned}
\|EQ\| = \sqrt{(5+tp)^2 + (5-10t)^2} &= 7\sqrt{2} \\
25 + t^2p^2 + 10tp + 25 - 100t + 100t^2 &= 98 \\
t^2(p^2+100) + 10tp - 100t &= 48 \\
100t + 10tp - 100t &= 48 \\
10tp = \left(\frac{1000}{p^2+100}\right)p &= 48 \\
1000p &= 48(p^2+100) \\
48p^2 - 1000p + 4800 &= 0 \\
6p^2 - 125p + 600 = (3p-40)(2p-15) &= 0 \\
p &= \frac{40}{3} \quad \text{or} \quad p = \frac{15}{2}.
\end{aligned}$$

The distances from $Q = (10+tp, 10-10t)$ to $A = (10,10)$ and $D = (10,0)$ are

$$\begin{aligned}
\|AQ\| &= \sqrt{t^2p^2 + 100t^2} \\
&= t\sqrt{p^2+100} \\
&= \frac{100}{\sqrt{p^2+100}}
\end{aligned}$$

and

$$\begin{aligned}\|DQ\| &= \sqrt{t^2p^2 + (10 - 10t)^2} \\ &= \sqrt{t^2p^2 + 100 + 100t^2 - 200t} \\ &= \sqrt{t^2(p^2 + 100) + 100 - 200t} \\ &= \sqrt{100t + 100 - 200t} \\ &= 10\sqrt{1 - t} \\ &= 10\sqrt{\frac{p^2}{p^2 + 100}} \\ &= \frac{10p}{\sqrt{p^2 + 100}}.\end{aligned}$$

Hence,

$$|\| AQ \| - \| DQ \|| = \frac{10 \mid 10 - p \mid}{\sqrt{p^2 + 100}}.$$

Plugging $p = \frac{40}{3}$ yields

$$\mid 10 - p \mid = \frac{10}{3} \quad \text{and} \quad \sqrt{p^2 + 100} = \frac{50}{3},$$

so that

$$|\| AQ \| - \| DQ \|| = \frac{10 \cdot \frac{10}{3}}{\frac{50}{3}} = \frac{100}{50} = \boxed{2}.$$

Similarly, one can plug in $p = \frac{15}{2}$ and get the same answer.

10. Let $AFDC$ be a rectangle. Construct points E and B outside of $AFDC$ such that $AB = BC = DE = EF = 45$ and $ABCDEF$ is a convex hexagon. Let $\mathcal{E}$ be an inscribed ellipse tangent to sides AB, BC, CD, DE, EF, FA at points U, V, W, X, Y, and Z, respectively. Points F_1 and F_2 with F_1 closer to B are the foci of $\mathcal{E}$ satisfying $\triangle ABC \cong \triangle F_1WF_2 \cong \triangle DEF$. Let Q be on line CD such that $F_1Y \perp ZQ$. Compute the area of quadrilateral F_1QYZ.

    Answer: $1350\sqrt{3}$

    Solution: First, we solve this problem when $AB = BC = DE = EF = 1$, then scale by 2025 at the end.

    Lemma 1. F_1ZF_2W is a square.

    Proof: Since ZW is parallel to AC and F_1F_2 is perpendicular to AC, F_1ZF_2W has perpendicular diagonals. By symmetry, these diagonals bisect each other as well, so F_1ZF_2W is a rhombus. However, $F_1F_2 = AC = ZW$, so these diagonals bisect each other; hence, F_1ZF_2W is a square.

    From this we can conclude that $\triangle ABC \cong \triangle F_1WF_2 \cong \triangle DEF$ are all $45 - 45 - 90$ triangles.

    Lemma 2. $\angle YF_1E = 15^\circ$.

Proof: By the previous lemma, $\angle DEF = 90°$. Line F_1F_2 bisects this, so $\angle F_1EY = \angle F_2EY = 45°$.

By a well-known property of ellipses, as F_1 and F_2 are foci of the ellipse and Y is the tangency point of this ellipse to segment EF, we have $\angle F_1YF = \angle F_2YE$. Consider the point F_3 constructed by reflecting F_2 across segment EF; it thus follows that F_1, Y, and F_3 are collinear. But simultaneously, $\angle F_2EF_3 = 2\angle F_2EY = 90°$. Hence, $\triangle F_1EF_3$ is a right triangle with a right angle at E.

Let $x = F_3E$; we find the side lengths of $\triangle F_1EF_3$ in terms of x. First,

$$F_1F_3 = F_1Y + YF_3 = F_1Y + YF_2.$$

Since the sum of the distances from any point on an ellipse to its foci is constant, this is equal to $F_1Z + ZF_2 = 2$. Meanwhile,

$$F_1E = F_1F_2 + F_2E = \sqrt{2} + x$$

and $F_3E = x$.

By the Pythagorean Theorem, $(\sqrt{2} + x)^2 + x^2 = 4$, and solving, $x = \frac{\sqrt{6}-\sqrt{2}}{2}$. Then,

$$\sin(\angle YF_1E) = \sin(\angle F_3F_1E) = \frac{\sqrt{6} - \sqrt{2}}{4} = \sin 15°$$

so $\angle YF_1E = 15°$ as desired.

These suffice to compute the lengths of the diagonals of F_1QYZ.

Since two angles of $\triangle F_1YE$ are $\angle YF_1E = 15°$ and $\angle F_1EY = 45°$, the remaining angle $\angle F_1YE$ must be $120°$. However, as stated previously,

$$\angle EYF_2 = \angle FYF_1 = 60°$$

so $\angle F_1YF_2 = 60°$. We can then use Law of Sines on $\triangle F_1YF_2$ to get

$$F_1Y = \frac{\sin(180° - 15° - 60°)}{\sin(60°)} \cdot \sqrt{2} = \frac{\sin(75°)}{\sin(60°)} \cdot \sqrt{2}.$$

Meanwhile, line F_1Y forms a $15°$ angle with line F_1E which is parallel to line CD, so line QZ, which is perpendicular to line F_1Y, must form a $75°$ angle with line CD. Hence $\angle ZQW = 75°$, so

$$QZ = \frac{ZW}{\sin(75°)} = \frac{\sqrt{2}}{\sin(75°)}.$$

Finally, the area of F_1QYZ is

$$\frac{F_1Y \cdot QZ}{2} = \frac{1}{\sin(60°)} = \frac{2\sqrt{3}}{3}$$

which scaled up by 2025 becomes $\boxed{1350\sqrt{3}}$.

TB. *This is an estimation question used for tiebreaking purposes. Ties on this test will be broken by absolute distance from the correct answer to this question.*

The incircle of triangle ABC is tangent to AB at F such that $AF = 5$, $BF = 12$, and the area of the triangle is 20250.

Estimate $\cos \angle ABC$ in the decimal form $0.abcdefgh$ where a, b, c, d, e, f, g, h are decimal digits each between 0 and 9, inclusive.

Answer: 0.41492712

Solution: We can compute it directly, albeit with a lot of calculation. Here is a way to approximate:
Let D and E be its tangency on BC and AC respectively. By the tangency condition, $AE = AF = 5$, $BD = BF = 12$, $CD = CE$. As such, $CB - CA = (CD + DB) - (CE - EA) = (CD - CE) + (DB - EA) = 12 - 5 = 7$ is an invariant independent of location of C. This implies that the curve is half of a hyperbola with focuses at A and B, passing through F. $\angle ABC$ approaches its upper bound as C approaches the hyperbola's point at infinity, where CA and CB are parallel. Let H be the projection of A on to BD. $BH = CB - CA = 7$. Since $\angle ABC = \angle ABH$, $AB = 12 + 5 = 17$, and $\angle AHB$ is a right angle, $\cos \angle ABC = \cos \angle AHB = \frac{BH}{AB} = \frac{7}{17} \approx \boxed{0.41}$.

Here is a direct computation: We are given that the incircle of triangle ABC is tangent to side AB at point F, with $AF = 5$, $BF = 12$, and area of triangle is 20250.

Let $AB = c = 17$, since $AF + BF = 5 + 12 = 17$. Let s be the semi-perimeter of triangle ABC, and let the side lengths be:

$$a = BC = s - 5, \quad b = AC = s - 12, \quad c = AB = 17$$

The area can be written as:

$$\text{Area} = r \cdot s = 20250$$

Using Heron's formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s \cdot 5 \cdot 12 \cdot (s-17)} \Rightarrow 20250 = \sqrt{60s(s-17)}$$

Square both sides:

$$20250^2 = 60s(s-17) \Rightarrow s(s-17) = \frac{20250^2}{60} = 6834375$$

Solve the quadratic:

$$s^2 - 17s - 6834375 = 0 \Rightarrow s = \frac{17 + \sqrt{27337789}}{2} = \frac{17 + 5229}{2} = 2623$$

Then:

$$a = s - 5 = 2618, \quad b = s - 12 = 2611$$

Use the Law of Cosines at angle $\angle ABC$:

$$\cos\angle ABC = \frac{a^2 + c^2 - b^2}{2ac}$$

Using expressions:

$$\cos\angle ABC = \frac{(s-5)^2 + 17^2 - (s-12)^2}{2\cdot(s-5)\cdot 17} = \frac{14s+170}{34(s-5)} = \frac{36892}{88912} = \frac{9223}{22228}$$

$$\cos\angle ABC = \boxed{\tfrac{9223}{22228} = 0.4149271189...}.$$