1. Your team is participating in the Guts round! You see a problem which consists of 5 Yes/No questions and peculiar scoring. You will submit n of the 5 problems and receive $n(n-1)$ points if they're all right and 0 points if any of them is wrong. Suppose your team is 100% certain about the answers for two of the questions and has no clue for the other three (50% chance of getting each right). Compute the average number of points your team will get if they submit answers to the questions to maximize this average.

2. A rectangle with integer side lengths has a diagonal with the same length as a diagonal of a square with an area of 37. Compute the area of the rectangle.

3. A jar has only green marbles and orange marbles. If half of the orange marbles were removed, the percentage of marbles in the jar that are green would be 81.25%. If half of the green marbles were removed instead, the percentage of marbles in the jar that are orange would be $n\%$. Compute n.

4. The nonzero digits A, B, and C are chosen such that the three-digit number $\underline{A}\,\underline{B}\,\underline{C}$ is a multiple of 25, and the three-digit number $\underline{C}\,\underline{B}\,\underline{A}$ is a multiple of 36. Compute the product of the three digits, that is $A \times B \times C$.

5. An integer n between 1 and 100, inclusive, is selected uniformly at random. Alina computes the remainder when n is divided by 6, and Laila computes the remainder when n is divided by 4. Compute the probability that Alina's remainder is strictly greater than Laila's remainder.

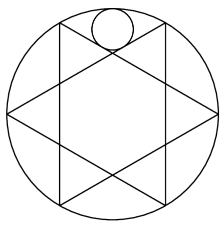
6. The closed shaded region in the image is made of semicircular arcs and parallel horizontal line segments. Each of the line segments has length 10, and adjacent line segments are spaced 1 unit apart from each other. Compute the area of the shaded region.

7. For a positive integer n, let $s(n)$ denote the sum of digits of n. Compute the number of integers $1 \le n \le 45$ such that $s(n) = s(n^2)$.

8. Compute the unique integer $n > 7$ such that $\frac{8^n-1}{7}$ expressed in binary has the same number of digits as $\frac{81^{n-7}-1}{8}$ expressed in base three.

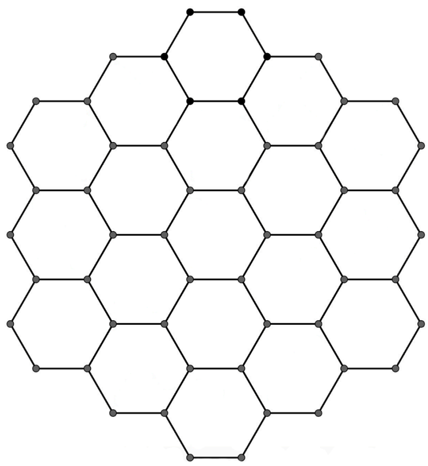
9. Compute the sum of all primes p for which there exists an integer x such that $72 + 5x^{p-1}$ is divisible by p.

10. In the diagram to the right, the two equilateral triangles intersect to form a regular hexagon. Circle ω is tangent to both equilateral triangles and internally tangent to the circumcircle Ω of both triangles. If the radius of Ω is 1, compute the radius of ω.



11. Aristotle is thinking of a monic quadratic polynomial $P(x) = x^2 + ax + b$ with roots r and s. He tells Plato r and a and tells Socrates r and b. However, Plato mistakenly believes that a is the constant term, and Socrates believes that b is the coefficient of x. Plato and Socrates independently compute s, and their (incorrect) results multiply to 1. If $s = 5$, compute r.

12. In the grid to the right, you want to color as many hexagons red as possible. However, no three adjacent hexagons whose centers lie on the same line can be colored red. Compute the maximum number of hexagons you can color red.



13. A sequence of real numbers $\{a_n\}$ satisfies $a_0 = \frac{\sqrt{3}}{3}$ and $a_n = \frac{1}{a_0}\sqrt{\sum_{k=0}^{n-1} a_k^2}$ for $n \geq 1$. Define the sequence $\{b_N\}$ by

$$b_N = \frac{\sum_{n=1}^{2N} a_n}{\sum_{n=1}^{N} a_n}$$

for $N \geq 1$. Compute the smallest value of N such that $b_N \geq 2025$.

14. Sophie has a right triangle T with sides 6, 8, and 10. Let $\mathcal{P}$ be the set of all points P within T such that there is a circle of radius 1 entirely contained within T with P lying on its circumference. Compute the area of $\mathcal{P}$.

15. Define a function $f(n)$ by

$$f(n) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor,$$

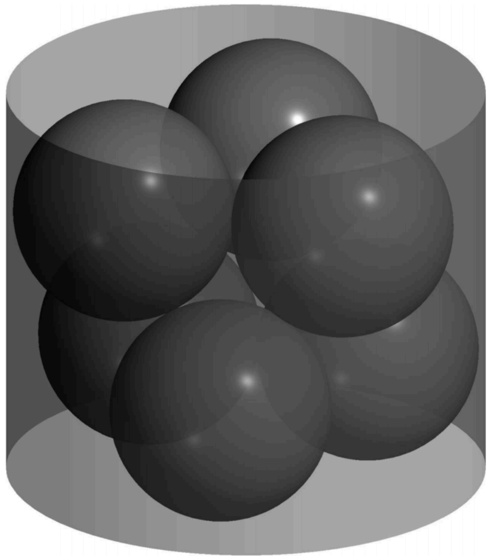
where n is a positive integer. Compute the number of integers n between 1 and 2025, inclusive, such that $f(n)$ is odd.

16. Compute the number of sets of 10 distinct positive integers $S = \{a_1, \ldots, a_{10}\}$ with the following property: the sum of all integers in S is at most 1024, and every nonempty subset of S has a distinct sum.

17. Let $h(n)$ represent the number of ones in the binary representation of n. Compute $\sum_{k=1}^{2025} h(k)$.

18. Find the largest of the three primes dividing 20250000002401.

19. Six spheres of radius 1 are placed inside a cylinder (with bases parallel to the ground) in two layers, as shown in the image. Each layer consists of three spheres all tangent to each other, and each sphere in the top layer is tangent to two spheres in the bottom layer. The cylinder's wall is tangent to all six spheres, and each of its bases is tangent to the three spheres in one layer. Compute the height of the cylinder.



20. Given a positive integer, your job is to make it into 1 by a series of operations. At each step you can do one of these two actions:

    1. If the number has at least two distinct prime divisors, replace the number with its second smallest distinct prime divisor.
    2. Subtract 1.

    Let $f(n)$ be the minimum number of steps required to make n into 1 using these operations. Compute $\max_{n\geq 1} f(n)$; that is, the maximum value of $f(n)$ over all positive integers n.

21. Let ω_1 and ω_2 be circles with centers O_1 and O_2, and radii 2 and 3, respectively. Let ω_3 be a third circle centered at O_3, intersecting ω_1 at point X, ω_2 at point Y, such that $O_1X \perp XO_3$ and $O_2Y \perp YO_3$. Additionally, ω_3 intersects O_1O_2 at distinct points M and N, and $O_1O_2 = 7$. Over all possible circles ω_3, compute the maximum possible value of MN.

22. Call an ordered quintuple (p, q, r, s, t) of integers *good* if $-3 \leq p, q, r, s, t \leq 3$, and there exists a nonzero solution $(a, b, c, d, e) \neq (0, 0, 0, 0, 0)$ to the following system of equations:

$$\begin{cases} a + bp + cp^2 + dp^3 + ep^5 = 0 \\ a + bq + cq^2 + dq^3 + eq^5 = 0 \\ a + br + cr^2 + dr^3 + er^5 = 0 \\ a + bs + cs^2 + ds^3 + es^5 = 0 \\ a + bt + ct^2 + dt^3 + et^5 = 0. \end{cases}$$

    Here, a, b, c, d, e are real numbers. Compute the number of *good* quintuples.

23. Nine giraffes labeled 1 through 9 are running around a circular track, all starting at the same position called "position zero." Giraffe 1 runs 1 mile per hour. For $2 \leq i \leq 9$, giraffe i runs $1 + \frac{i}{10}$ miles per hour. At the start of the race, each giraffe decides to run the track clockwise or counterclockwise at random. Compute the expected value of the label of the first giraffe that giraffe 1 meets after the race begins.

24. Let S be the set of positive divisors of $N = 2025^{2025}$. A subset M of S is *maximally inconceivable* if it has the following properties: (1) $|M| \geq 2$, (2) for any two elements a, b in M, $15 \mid \frac{\operatorname{lcm}(a,b)}{\gcd(a,b)}$, and (3) there does not exist an element in S not in M that can be added to M while preserving property (2). Compute the number of possible values of $\gcd(M)$

over all maximally inconceivable subsets M, where $\gcd(M)$ is the greatest common divisor of all of the elements in M.

25. Let I and O be the incenter and circumcenter of $\triangle ABC$, respectively, and let ω be its incircle. Suppose that the tangents from O to ω meet ω at E and F, and suppose EF meets BC at D. If the inradius of $\triangle ABC$ is 2 and $OI = OD = \sqrt{10}$, find BC^2. (There are two possible configurations. Either answer will be accepted.)

26. Call a permutation σ of $\{1, 2, ..., 9\}$ *almost-convex* if there exists at most one triple $a, b, c \in \{1, 2, ..., 9\}$ such that $a < b < c$, and

$$\sigma(b) > \frac{\sigma(c) - \sigma(a)}{c - a}(b - a) + \sigma(a).$$

Compute the number of almost-convex permutations of $\{1, 2, ..., 9\}$.

27. We define the m-*expansion* of a multiset of numbers (a multiset is a set where duplicates are allowed) as the maximum sum of squares among all of the subsets of size m (formally, the expansion is $\max_{S: |S| = m} \sum_{i \in S} a_i^2$).

Suppose that $a_1, a_2, ..., a_{170}$ are real numbers such that $\sum_{i=1}^{170} a_i^2 = 170$ and $\sum_{i=1}^{170} a_i^4 = 510$. Compute the smallest possible 85-expansion of any set $a_1, a_2, ..., a_{170}$ satisfying these properties.

28. A function $f : \{1, 2, ..., 100\} \to \{1, 2, ..., 100\}$ is constructed randomly (that is, each $f(i)$ for $1 \leq i \leq 100$ is selected from $\{1, 2, ..., 100\}$, uniformly and at random). For $2 \leq k \leq 100$, a length k *cycle* is a sequence of numbers $n_1, n_2, ..., n_k$ such that $f(n_i) = n_{i+1}$ for all $1 \leq i \leq k - 1$ and $f(n_k) = n_1$. A length 1 *cycle* is a number n satisfying $f(n) = n$. Note that a number is part of at most one cycle. Moreover, a number can be part of none: for example, if $f(1) = 2$ and $f(2) = 2$, then 1 is part of no cycles. Compute the expected sum of squares of the cycle lengths of f (including multiplicity, and each cycle of f is counted exactly once).

29. Welcome to the **USAYNO** (USA Yes/No Olympiad), a contest where every question has a yes/no answer! Each of the next four problems consists of 5 yes/no questions. For each of these, if you answer n questions and get them all correct, you will receive $n(n-1)$ points. If *any* of your answers for a given problem are incorrect, you will receive 0 points.

Submit your answer as a string of 5 letters, with $\boldsymbol{Y}$ representing Yes, $\boldsymbol{N}$ representing No, and $\boldsymbol{B}$ representing Blank (no submission). An example string is **BYBNY**.

1. An *algebraic number* is any real number r which is the root of some polynomial with integer coefficients. A *bi-algebraic number* is any real number r which is a root of some polynomial with coefficients being algebraic numbers. Are there numbers which are bi-algebraic but not algebraic?

2. Call a function k-periodic if $f(x) = f(x + k)$ for all real numbers x. Does there exist a series of functions $f_1, f_2, ...$ each with integer period such that $\sum_{i=1}^{\infty} f(x) = x$ for all *irrational* $x > 0$?

3. Does there exist a complex number z such that $|\operatorname{Im}(z^n)|$ is strictly increasing with n?

4. Let A be the area bounded between the x-axis, y-axis, $y = e^{\cos x^2}$, and $x = \sqrt{\frac{17\pi}{4}}$; and let B be the area bounded between the x-axis, y-axis, $y = e^{\sin x^2}$, and $x = \sqrt{\frac{17\pi}{4}}$. Is $A > B$?

5. Do there exist three positive real numbers α, β, γ such that each positive integer can be represented as exactly one of $\lfloor k\alpha \rfloor$, $\lfloor k\beta \rfloor$, $\lfloor k\gamma \rfloor$ for some positive integer k?

30. 1. Consider a chessboard which is an 8×8 grid of squares. A king is a chess piece which can move uniformly at random to any non-blocked position adjacent (either orthogonally or diagonally) to its current position until it returns to its original starting point. Is it possible to place the king on the chessboard and block off certain squares in such a way that the expected time until the king returns to its starting location is exactly $\frac{64}{9}$?

2. Is it possible to color every point on the 2D plane (that is, $\mathbb{R}^2$) with finitely many colors such that any two points on the plane that are a positive integer distance apart have different colors?

3. A class of 10 people form clubs, each consisting of 6 distinct people. Is it possible to create 120 such clubs such that no two clubs are exactly the same or have exactly 3 people in common?

4. There are 2025 people, each of whom is given a black or red hat with equal (50-50) probability. Each person cannot see their own hat, but can see each of the other people's hats. Each person either chooses to guess their hat color or say nothing, and the group wins if at least one person guesses correctly and nobody guesses incorrectly. After the group receives their hats, aside from stating their guesses or saying nothing, the people are not allowed to communicate with each other. Is there a strategy under which the people win with probability at least $\frac{3}{4}$?

5. You are given a set $\mathcal{S}$ of *nonempty* subsets of $\{1, 2, ..., 2025\}$ such that if $A, B \in \mathcal{S}$, then $A \cup B \in \mathcal{S}$, and there is some subset A in $\mathcal{S}$ of size at most 2. Does there always exist some $1 \leq i \leq 2025$ contained in at least half of the subsets in $\mathcal{S}$?

31. 1. Given a square piece of paper, is it possible to (using a straightedge, compass, and by folding the piece of paper) construct three points A, B, and C such that $\frac{|AB|}{|BC|} = \sqrt[3]{2}$?

2. Let S be the sphere in $\mathbb{R}^3$ with equation $x^2 + y^2 + z^2 = 1$. Is the surface area of the portion of S lying on or above the plane $z \geq \frac{1}{2}$ less than π?

3. Given the plane ($\mathbb{R}^2$) initially colored white, you have two drawing implements: a pencil'', which in one move colors a unit disk (including its boundary) black, and an eraser'', which in one move colors a unit disk (including its boundary) white. With just these two implements and a possibly infinite number of moves, is it possible to make it so the only black part of the plane is a filled in regular pentagon (including its boundary) with side length $\frac{1}{5}$?

4. Let $O = (0, 0)$, $A = (1, 0)$, and $B = (0, 1)$. For any C on OA, let D be the point on OB such that $OC = BD$ and $AC = OB$, and let S be the area swept out by CD over all choices of C. Is S larger than $1 - \frac{\pi}{4}$?

5. Consider placing 4 distinct points in a plane such that no three of them lie on line, and the edges connecting each pair of vertices don't intersect in their interiors. Is it possible to place these points in such a way that every edge has integer length?

32. 1. Are $n^3 + 12$ and $(n + 1)^3 + 12$ relatively prime for all integers n?

2. Define the 2-parse of a positive integer as a way to split its decimal digits into contiguous segments where each segment represents a power of 2. For example, the integer 23221 has the 2-parse $[2, 32, 2, 1]$, because 2, 32, 2, and 1 are all powers of 2. Does there exist a positive integer with more than one distinct 2-parse?

3. Does there exist a prime with more than two distinct digits such that any permutation of its digits yields a prime? Disregarding the distinctness constraint, a two-digit example is 13, since 31 is also prime.

4. Every positive integer can be written as the sum of at most 9 positive integer cubes. Let $f(n)$ be the minimum number of positive integer cubes whose sum is n. Among the integers $1 \leq n \leq 2025$, are there more values of n for which $f(n) = 3$ than for which $f(n) = 6$?

5. Let a and b be distinct positive integers whose digits contain no zeroes. Define the infinite string s formed by the concatenation $baaaa\cdots$ and interpret each prefix of s as a base-10 integer. A prime p is good'' if p divides at least one prefix of s. Does there exist a, b such that infinitely many primes are good and infinitely many primes are not good?