1. Your team is participating in the Guts round! You see a problem which consists of 5 Yes/No questions and peculiar scoring. You will submit n of the 5 problems and receive $n(n-1)$ points if they're all right and 0 points if any of them is wrong. Suppose your team is 100% certain about the answers for two of the questions and has no clue for the other three (50% chance of getting each right). Compute the average number of points your team will get if they submit answers to the questions to maximize this average.

   Answer: 3

   Solution: If you submit 2 problems, you will get 2 points with probability 1. If you submit 3 problems, you will get 6 points with probability $\frac{1}{2}$ (so expectation 3). If you submit 4 problems, you will get 12 points with probability $\frac{1}{4}$ (so expectation 3). If you submit 5 problems, you will get 20 points with probability $\frac{1}{8}$ (so expectation 2.5). Therefore, you should submit either 3 or 4 problems and get $\boxed{3}$ points.

2. A rectangle with integer side lengths has a diagonal with the same length as a diagonal of a square with an area of 37. Compute the area of the rectangle.

   Answer: 35

   Solution: The side length of a square with area 37 is $\sqrt{37}$, so the length of its diagonal is $\sqrt{37} \cdot \sqrt{2} = \sqrt{74}$. Letting the length and width of the rectangle be a and b, respectively, we have $a^2 + b^2 = \sqrt{74}^2 = 74$ (by the Pythagorean Theorem). Noting that $8^2 < 74 < 9^2$, it suffices to try values $b = 1, 2, 3, 4, 5, 6, 7, 8$ and see which give integer-valued a. This gives $a = 7, b = 5$ (or $a = 5, b = 7$) as the only solutions, since $7^2 + 5^2 = 49 + 25 = 74$, with no other values possible. The area of the rectangle is thus $7 \cdot 5 = \boxed{35}$.

3. A jar has only green marbles and orange marbles. If half of the orange marbles were removed, the percentage of marbles in the jar that are green would be 81.25%. If half of the green marbles were removed instead, the percentage of marbles in the jar that are orange would be $n\%$. Compute n.

   Answer: 48

   Solution: Let x and y be the number of green marbles and orange marbles initially in the jar, respectively. We are given the ratio relationship

$$\frac{x}{x + \frac{y}{2}} = \frac{81.25}{100} = \frac{13}{16}.$$

   This means that the green-to-total ratio is $13 : 16$ after halving the orange marbles. The new green-to-orange ratio, in this case, is then $13 : (16 - 13) = 13 : 3$. The original green-to-orange ratio, before halving the orange marbles, is therefore $13 : (3 \cdot 2) = 13 : 6$. If half of the green marbles were removed, green-to-orange ratio becomes $\frac{13}{2} : 6 = 13 : 12$, and thus the orange-to-total becomes $12 : (12 + 13) = 12 : 25$. The percentage $n\%$ is $\dfrac{12}{25} \cdot 100 = \boxed{48}\%$.

4. The nonzero digits A, B, and C are chosen such that the three-digit number $\underline{A}\,\underline{B}\,\underline{C}$ is a multiple of 25, and the three-digit number $\underline{C}\,\underline{B}\,\underline{A}$ is a multiple of 36. Compute the product of the three digits, that is $A \times B \times C$.

   Answer: 210

Solution: Since $\underline{A}\,\underline{B}\,\underline{C}$ is a multiple of 5 and C is nonzero, $C = 5$. In order for $\underline{A}\,\underline{B}\,\underline{C}$ to be a multiple of 25, B must be equal to either 2 or 7. If $B = 2$, then for $\underline{C}\,\underline{B}\,\underline{A}$ to be a multiple of 36, it must be a multiple of 9, so

$$A + B + C \equiv A + 7 \equiv 0 \pmod 9 \implies A \equiv 2 \pmod 9 \implies A = 2.$$

This would make $\underline{C}\,\underline{B}\,\underline{A} = 522$ not be a multiple of 36. If $\underline{B} = 7$, then in order to make $\underline{C}\,\underline{B}\,\underline{A}$ a multiple of 36, it must be a multiple of 9, so

$$A + B + C \equiv A + 3 \equiv 0 \pmod 9 \implies A \equiv 6 \pmod 9 \implies A = 6.$$

We confirm that $\underline{C}\,\underline{B}\,\underline{A} = 576$ is a multiple of 36, so these digits satisfy the conditions of the problem! Therefore, the answer is $\underline{A} \times \underline{B} \times \underline{C} = 6 \times 7 \times 5 = \boxed{210}$.

5. An integer n between 1 and 100, inclusive, is selected uniformly at random. Alina computes the remainder when n is divided by 6, and Laila computes the remainder when n is divided by 4. Compute the probability that Alina's remainder is strictly greater than Laila's remainder.

   Answer: $\frac{49}{100}$

   Solution: The remainders loop every $\gcd(6, 4) = 12$ terms by the Chinese Remainder Theorem. We can find a pattern by computing the remainders for the first 12 numbers:

$$1 - 12 \bmod 6 : 1, 2, 3, 4, 5, 0, 1, 2, 3, 4, 5, 0$$
$$1 - 12 \bmod 4 : 1, 2, 3, 0, 1, 2, 3, 0, 1, 2, 3, 0$$

   For every 12 terms, 6 of the terms satisfy the property that Alina's remainder is greater than Laila's remainder. Among the first $96 = 12 \cdot 8$ positive integers, $12 \cdot 4 = 48$ of them are good. We then check that among $97, 98, 99$, and 100, only 100 satisfies the desired property. Therefore, there are 49 successful outcomes of 100 total outcomes, for a probability of $\boxed{\frac{49}{100}}$.

6. The closed shaded region in the image is made of semicircular arcs and parallel horizontal line segments. Each of the line segments has length 10, and adjacent line segments are spaced 1 unit apart from each other. Compute the area of the shaded region.

   Answer: $40 + \frac{25\pi}{4}$

   Solution: Note that if we take the curved regions from the right side of the region not including the small circle in the bottom-right and move them into corresponding empty region on the left side, we complete a semicircle on the left with radius $\frac{7}{2}$ and thus area $\frac{49\pi}{8}$. The remaining small semicircle in the bottom-right has radius $\frac{1}{2}$ and thus area $\frac{\pi}{8}$. The rectangular regions have total area $4 \cdot 10 = 40$. Hence, the total area of the shaded region is $40 + \frac{49\pi}{8} + \frac{\pi}{8} = \boxed{40 + \frac{25\pi}{4}}$.

7. For a positive integer n, let $s(n)$ denote the sum of digits of n. Compute the number of integers $1 \le n \le 45$ such that $s(n) = s(n^2)$.

   Answer: 6

   Solution: We know that, by the divisibility rule for 9, $n \equiv s(n) \bmod 9$. This implies that $n \equiv n^2 \bmod 9$. We check to see that the only solutions for this are $n \equiv 0, 1 \bmod 9$.

Checking over all numbers between 1 and 45 with remainders 0 or 1 modulo 9, we get that the possible values of n are $1, 9, 10, 18, 19$, and 45 for a total of $\boxed{6}$ integers.

8. Compute the unique integer $n > 7$ such that $\frac{8^n-1}{7}$ expressed in binary has the same number of digits as $\frac{81^{n-7}-1}{8}$ expressed in base three.

Answer: 27

Solution: Both numbers are the sum of geometric series. Specifically,

$$\frac{8^n-1}{7} = \frac{8^n-1}{8-1} = 8^{n-1} + 8^{n-2} + \cdots + 8^0 = 100100\cdots1001_2,$$

where there are a total of n ones and $2(n-1)$ zeroes. Furthermore,

$$\frac{81^{n-7}-1}{8} = \frac{9^{2(n-7)}-1}{9-1} = 9^{2n-15} + 9^{2n-14} + \ldots + 9^0 = 10101\cdots01_3,$$

where there are a total of $2n-14$ ones and $2n-15$ zeroes. Hence,

$$2n - 15 + 2n - 14 = n + 2(n-1) \implies n = \boxed{27}.$$

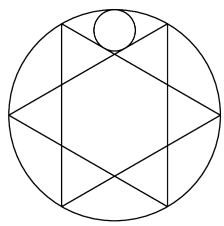
Note that $27 > 7$, so $\frac{8^n-1}{7}$ and $\frac{81^{n-7}-1}{8}$ are positive integers.

9. Compute the sum of all primes p for which there exists an integer x such that $72 + 5x^{p-1}$ is divisible by p.

Answer: 23

Solution: First, suppose p is valid; that is, there exists an integer x such that $72 + 5x^{p-1}$ is divisible by p. Fermat's Little Theorem inspires two cases. If x is divisible by p, then $72 + 5x^{p-1} \equiv 72 \equiv 0 \bmod p$, so p divides 72. The prime divisors of 72 are 2 and 3, so p can be 2 or 3. Otherwise, by Fermat's Little Theorem, we have $x^{p-1} \equiv 1 \bmod p$, implying $77 \equiv 0 \bmod p$, and p divides 77. The prime divisors of 77 are 7 and 11, so p can also be 7 or 11. Thus, the possible primes p are $2, 3, 7$, and 11. These work for $x = 2, 3, 1$, and 1, respectively. Thus our answer is $2 + 3 + 7 + 11 = \boxed{23}$.

10. In the diagram to the right, the two equilateral triangles intersect to form a regular hexagon. Circle ω is tangent to both equilateral triangles and internally tangent to the circumcircle Ω of both triangles. If the radius of Ω is 1, compute the radius of ω.



Answer: $3\sqrt{3} - 5$.

Solution: Draw the radius of Ω passing through the center of ω, and draw another radius ℓ at a 60 degree angle to that, going through an adjacent intersection of triangles. Drop an altitude from the center of ω to ℓ (intersecting at A) and look at the 30-60-90 triangle $O_\Omega O_\omega A$. The hypotenuse is $O_\Omega O_\omega = 1 - r$, and the longer leg is $O_\omega A = \frac{1}{2} + r$, since the distance from a side of the equilateral triangle to O_Ω is $\frac{1}{2}$.

Since this is a 30-60-90 triangle, we have that $\frac{\frac{1}{2}+r}{1-r} = \frac{\sqrt{3}}{2}$. Rearranging and solving for r gives $r = \boxed{3\sqrt{3} - 5}$.

11. Aristotle is thinking of a monic quadratic polynomial $P(x) = x^2 + ax + b$ with roots r and s. He tells Plato r and a and tells Socrates r and b. However, Plato mistakenly believes that a is the constant term, and Socrates believes that b is the coefficient of x. Plato and Socrates independently compute s, and their (incorrect) results multiply to 1. If $s = 5$, compute r.

Answer: $-\frac{29}{6}$

Solution: We're given that $P(x)$ has roots r and 5, so $a = -5 - r$ and $b = 5r$ by Vieta's formulas. Denote Plato and Socrates' incorrect values for s as m and n respectively. Plato believes that $-5 - r$ is the constant term, so

$$rm = -5 - r \implies m = -\frac{5}{r} - 1.$$

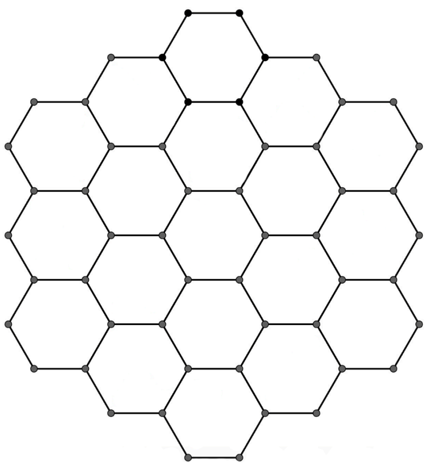
Socrates believes that $5r$ is the coefficient of x, so

$$-r - n = 5r \implies n = -6r.$$

We're given that $mn = 1$, so

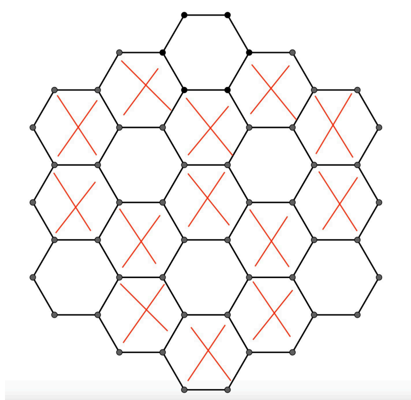
$$(-6r)\left(-\frac{5}{r} - 1\right) = 30 + 6r = 1 \implies r = \boxed{-\frac{29}{6}}.$$

12. In the grid to the right, you want to color as many hexagons red as possible. However, no three adjacent hexagons whose centers lie on the same line can be colored red. Compute the maximum number of hexagons you can color red.



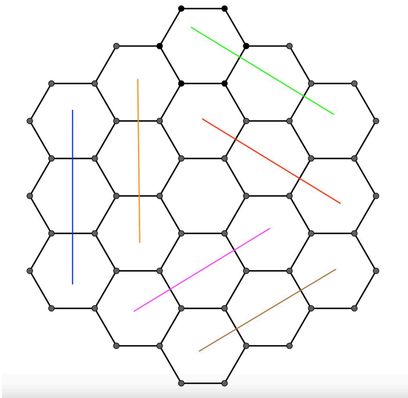
Answer: 13.

Solution: You can color all but 6 hexagons in the following way:



This gives a total of $19 - 6 = \boxed{13}$ colored hexagons.

To prove that this is the maximum number of colored hexagons, note that it is possible to find 6 non-overlapping 3-in-a-rows in the grid, so there must be at least 6 non-colored hexagons:



13. A sequence of real numbers $\{a_n\}$ satisfies $a_0 = \frac{\sqrt{3}}{3}$ and $a_n = \frac{1}{a_0}\sqrt{\sum_{k=0}^{n-1} a_k^2}$ for $n \geq 1$. Define the sequence $\{b_N\}$ by

$$b_N = \frac{\sum_{n=1}^{2N} a_n}{\sum_{n=1}^{N} a_n}$$

for $N \geq 1$. Compute the smallest value of N such that $b_N \geq 2025$.

Answer: 11

Solution: We claim that $a_n = 2^{n-1}$ for all $n \geq 1$. We will prove this using strong induction on n. For the base case, we have that

$$\begin{aligned} a_1 &= \frac{1}{a_0}\sqrt{a_0^2} \\ &= 1. \end{aligned}$$

Now suppose $a_k = 2^{k-1}$ for all $k \in \{1, \ldots, n-1\}$. Then

$$\begin{aligned} a_n &= \frac{1}{a_0}\sqrt{\sum_{k=0}^{n-1} a_k^2} \\ &= \frac{1}{a_0}\sqrt{a_0^2 + \sum_{k=1}^{n-1} 2^{2k-2}} \\ &= \sqrt{3} \cdot \sqrt{\frac{1}{3} + \frac{1-2^{2n-2}}{1-2^2}} \\ &= \sqrt{3} \cdot \frac{2^{n-1}}{\sqrt{3}} \\ &= 2^{n-1}. \end{aligned}$$

Therefore,

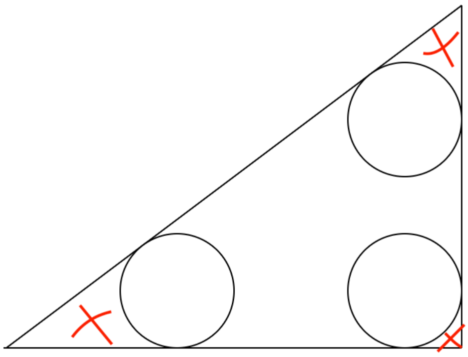
$$b_N = \frac{\sum_{n=1}^{2N} a_n}{\sum_{n=1}^{N} a_n} = \frac{\frac{1-2^{2N}}{1-2}}{\frac{1-2^N}{1-2}} = \frac{1-2^{2N}}{1-2^N} = 1 + 2^N.$$

We wish to find N such that $1 + 2^N \geq 2025$. If $N = 10$, $1 + 2^N = 1025 < 2025$, and if $N = 11$, $1 + 2^N = 2049 > 2025$. Therefore, $N = \boxed{11}$.

14. Sophie has a right triangle T with sides 6, 8, and 10. Let $\mathcal{P}$ be the set of all points P within T such that there is a circle of radius 1 entirely contained within T with P lying on its circumference. Compute the area of $\mathcal{P}$.

Answer: $18 + \pi$

Solution: The only places P cannot be is in the red regions here:



Note that the inradius of a $6-8-10$ triangle is $r = \frac{6 \cdot 8}{6+8+10} = 2$. So, we can instead imagine constructing a 3-4-5 triangle surrounding the circle in the bottom left corner and from here we immediately (via similar triangles) see that the sum of areas of the red regions is $\frac{3 \cdot 4}{2} - \pi = 6 - \pi$. Thus, the answer is $24 - (6 - \pi) = \boxed{18 + \pi}$.

mitexbold(Note:) We can extend this. Consider any triangle with area A and perimeter p and we wish to compute this area for a circle of radius r. The inradius of the original triangle is $r' = \frac{2A}{p}$, so the sum of areas of the red regions is $\left(\frac{r}{r'}\right)^2 \cdot A - \pi r^2 = r^2 \left(\frac{p^2}{4A} - \pi\right)$ and the answer is $A - r^2 \left(\frac{p^2}{4A} - \pi\right)$.

15. Define a function $f(n)$ by

$$f(n) = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor,$$

where n is a positive integer. Compute the number of integers n between 1 and 2025, inclusive, such that $f(n)$ is odd.

Answer: 991

Solution: Note that for $n \geq 2$,

$$f(n) - f(n-1) = \sum_{i=1}^{n} \left(\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor \right).$$

When $i \mid n$, we have $\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = 1$. Otherwise, $\left\lfloor \frac{n}{i} \right\rfloor - \left\lfloor \frac{n-1}{i} \right\rfloor = 0$. Hence, $f(n) - f(n-1)$ is the number of divisors of n. Define $d(n)$ to be the number of divisors of n. Note that $f(1) = 1 = d(1)$. Therefore,

$$f(n) = d(n) + f(n-1) = d(n) + d(n-1) + f(n-2) = \cdots$$

$$= \sum_{i=2}^{n} d(i) + f(1) = \sum_{i=1}^{n} d(i).$$

Recall that $d(n)$ is odd if and only if n is a perfect square. Therefore, $f(n)$ is odd if and only if $(2k-1)^2 \leq n < (2k)^2$ for some k i.e. there are an odd number of positive perfect squares less than or equal to n. In this interval, there are $(2k)^2 - (2k-1)^2 = 4k^2 - 4k^2 + 4k - 1 = 4k - 1$ valid integers. As $44^2 < 2025 = 45^2$, the sets of integers we must consider consist of whole intervals $[(2k-1)^2, (2k)^2)$ as well as a single integer 2025. Excluding 2025 at first, we compute the total length of the intervals with endpoints up to 2024 to be

$$\sum_{i=1}^{22} (2k)^2 - (2k-1)^2 = \sum_{i=1}^{22} 4k - 1 = 990.$$

Hence there are 990 possible n from 0 to 2024. We add 1 to account for $n = 2025$ to get $\boxed{991}$.

16. Compute the number of sets of 10 distinct positive integers $S = \{a_1, ..., a_{10}\}$ with the following property: the sum of all integers in S is at most 1024, and every nonempty subset of S has a distinct sum.

Answer: 2

Solution: For such an S, let the integers in the set be $a_1 < a_2 < ... < a_{10}$. Since there are 1023 non-empty subsets of S, if the total sum of the integers in S is less than 1023, then two subsets of S would have the same sum by the Pigeonhole Principle. Then, S would not be valid, so we conclude that the total sum is at least 1023. But, since the total sum is at most 1024, then the only possibilities for the total sum are 1023 and 1024; we split the problem into these two cases.

If the sum is 1023, then note that we must in fact have a subset of each possible sum from 1 through 1023. We claim that this forces the sequence $\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512\}$ by induction. Obviously, the smallest element of our set must be 1. Suppose that the smallest k elements of S are the first k powers of 2. Then, these elements generate the sums $1, ..., 2^k - 1$ via their subsets. Our next smallest element then must be 2^k: if 2^k weren't in S, then there would be no subset of S with sum 2^k. This shows that in this case, $S = \{1, 2, ..., 512\}$.

Next, suppose that the sum is 1024. Then, notice that we must miss exactly one sum $1 \leq x \leq 1024$. Certainly, we can construct the set $\{1, 2, 4, 8, 16, 32, 64, 128, 256, 513\}$ which works. We prove now that this is the unique possibility for a sum of 1024. To show this, let's do casework on the missed sum.

Case 1: We skip the sum 1. This forces $a_1 = 2$. Remarkably, we claim that this is impossible. We can uniquely construct the smallest 9 elements of our set to be $\{2, 3, 4, 8, 16, 32, 64, 128, 256\}$ (similarly to the induction proof above, each number after 2 is forced as the smallest inexpressible number). However, since the total sum is 1024, our last element must be 511, which is expressible in terms of these first 9 elements (by adding except 2). Hence, this case is impossible.

Case 2: We skip some sum x between 2 and 1024. We claim that x has to be a power of 2. Consider if there were k elements in S smaller than x: these generate all the distinct sums from 1 to $x - 1$. But since there are k elements, there are $2^k - 1$ nonempty subsets of these elements, so $x - 1 = 2^k - 1$ and $x = 2^k$.

Clearly the set exemplified above is the only way to skip the single sum 512. Our remaining possibilities are thus the powers of 2 from 2^1 to $2^8 = 256$.

Suppose $x = 2^k$ for $2 \le k \le 8$. Then, our first k elements are $1, 2, ..., 2^{k-1}$. We skip the sum 2^k, so the next element after this is $2^k + 1$. Now, the sums in the interval $[2^k + 1, 2^{k+1}]$ are expressible, so the element after this is $2^{k+1} + 1$.

If $k \le 7$, we have at most 9 elements now. Our most recently added element allows us to express the sums in $[2^{k+1} + 1, 2^{k+1} + 2^k]$ but not $2^{k+1} + 2^k + 1$, so $2^{k+1} + 2^k + 1$ must be our next number. However, adding up our chosen elements $(2^k + 1) + (2^{k+1} + 1)$ and $(1) + (2^{k+1} + 2^k + 1)$ give the same result, so we have been forced into including elements without distinct subset sums. This is a contradiction, so such x are impossible to obtain.

If $k = 8$, then currently $S = \{1, 2, ..., 128, 257, 513\}$. However, this has a sum of 1025, which is invalid. Hence, this is impossible as well.

Therefore, our only valid S come from skipping the sums 1023 and 1024, each of which produce one possible S, so the answer is $\boxed{2}$.

17. Let $h(n)$ represent the number of ones in the binary representation of n. Compute $\sum_{k=1}^{2025} h(k)$.

Answer: 11067

Solution: Newest solution: First, we may compute that the total number of ones among all integers from 0 through 2047 is exactly $11 \cdot 2048 \cdot \frac{1}{2} = 11264$ (each bit is filled in exactly half of the numbers). Next, we have to subtract off the contribution from the numbers $2026, 2027, ..., 2047$. Note that $h(2047 - k) = 11 - h(k)$, so the amount to subtract is $22 \cdot 11 - \sum_{k=0}^{21} h(k)$. We see that $\sum_{k=0}^{15} h(k) = 4 \cdot 16 \cdot \frac{1}{2} = 32$ by a similar idea as the sum of everything from 0 to 2047. Now, we can just compute $h(16) = 1, h(17) = 2, h(18) = 2, h(19) = 3, h(20) = 2, h(21) = 3$ so the amount to subtract is

$$22 \cdot 11 - (32 + 1 + 2 + 2 + 3 + 2 + 3) = 242 - 45 = 197.$$

Therefore, the final answer is $11264 - 197 = \boxed{11067}$.

New solution: We are including every number non-negative integer smaller than $2^(11)$. Those are all the numbers represented by strings of 1 and 0 of length 11. On every position, there is a 50% chance of it being 1. In total we have $2048 \cdot 11$ digits, so $\frac{2048 \cdot 11}{2} = 11264$.

Original solution: Let $f(n)$ represent the number of ones used to represent all n-digit (in binary) numbers.
Then we have the recursive relationship $f(n) = 2f(n-1) + 2^n$.
Then it follows that $f(n) = n \cdot (2^{n-1})$.
Since $2^{10} < 2047 < 2^{11}$, it should be easy to see that all positive integers up to and including 2047 take 11 digits to represent in binary.
Therefore, it takes $f(11) = 11*2048 + 1 = 11265$ ones in total to represent 2048 also.

18. Find the largest of the three primes dividing 20250000002401.

Answer: 4479049

Solution: An important observation is that $2025 = 45^2$ and $2401 = 7^4$. Indeed,

$$\begin{aligned}
&20250000002401 \\
&= 45^2 \cdot 10^{10} + 7^4 \\
&= 2^{10} \cdot 3^4 \cdot 5^{12} + 7^4 \\
&= 2^2 \cdot (2^2 \cdot 3 \cdot 5^3)^4 + 7^4 \\
&= 4 \cdot 1500^4 + 7^4 \\
&= (7^2 - 3000 \cdot 7 + 2 \cdot 1500^2)(7^2 + 3000 \cdot 7 + 2 \cdot 1500^2)
\end{aligned}$$

by Sophie-Germaine identity. Let $a = 7^2 - 3000 \cdot 7 + 2 \cdot 1500^2$ and $b = 7^2 + 3000 \cdot 7 + 2 \cdot 1500^2$. Since the starting number has exactly three prime divisors, we just now need to check which one is not prime. When trying prime divisors, we can rule out easily $2, 3, 5, 7$. To check 11, we reduce modulo 11:

$$a = 7^2 - 3000 \cdot 7 + 2 \cdot 1500^2 \equiv 5 - 3 \cdot 4 + 2 \cdot 4^2 \equiv 3 \not\equiv 0 \pmod{11}$$

and

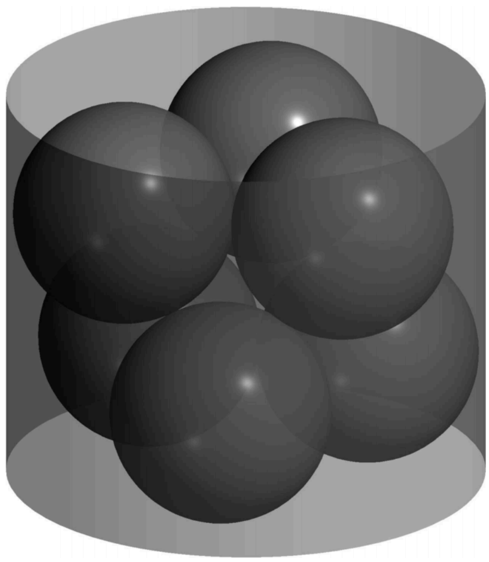
$$b = 7^2 + 3000 \cdot 7 + 2 \cdot 1500^2 \equiv 5 + 3 \cdot 4 + 2 \cdot 4^2 \equiv 5 \not\equiv 0 \pmod{11}.$$

We do the same thing for modulo 13:

$$b = 7^2 + 3000 \cdot 7 + 2 \cdot 1500^2 \equiv 10 + 10 \cdot 7 + 2 \cdot 5^2 \equiv 0 \pmod{13}.$$

As such, the three prime divisors of 20250000002401 are 13, $\frac{b}{13}$, and a. Clearly, a is the greatest prime here, and a is computed to be $\boxed{4479049}$.

19. Six spheres of radius 1 are placed inside a cylinder (with bases parallel to the ground) in two layers, as shown in the image. Each layer consists of three spheres all tangent to each other, and each sphere in the top layer is tangent to two spheres in the bottom layer. The cylinder's wall is tangent to all six spheres, and each of its bases is tangent to the three spheres in one layer. Compute the height of the cylinder.



Answer: $2 + \frac{2\sqrt{6}}{3}$

Solution: Note that the centers of the spheres form an octahedron with side length 2. The distance between the planes containing the centers of the top spheres and the centers of the bottom spheres thus is the distance between two opposite faces of the octahedron. We can divide the octahedron into eight right triangular pyramids to compute this; each pyramid has a height of $\sqrt{2}/\sqrt{3}$, so the height of the octahedron is $2\sqrt{2}/\sqrt{3}$. We add 2 for the radii of the top and bottom spheres above and below the octahedron. Hence, the height is $\boxed{2 + \frac{2\sqrt{6}}{3}}$.

20. Given a positive integer, your job is to make it into 1 by a series of operations. At each step you can do one of these two actions:

    1. If the number has at least two distinct prime divisors, replace the number with its second smallest distinct prime divisor.
    2. Subtract 1.

    Let $f(n)$ be the minimum number of steps required to make n into 1 using these operations. Compute $\max_{n \geq 1} f(n)$; that is, the maximum value of $f(n)$ over all positive integers n.

    Answer: 8

    Solution: First, let's establish some intuition. Given a big number, operation 1 will reduce is much more than operation 2. We would like to use 1 to achieve a small number, and then finish by series of 2.

    For that, we cannot start applying 1 to any n. Notice that if we have number divisible by 6, applying 1 will result in 3. This is very close, as it is enough to apply 2 twice to get 1.

    This leads us to a solution if $n \geq 6$:

    Step 1: Make our number divisible by 6 using operation 2.

    Step 2: Apply operation 1 resulting in 3.

    Step 3: Apply operation 2 twice to reach 1.

    Step 1 can take at most 5 operations. Step 2 takes 1 operation, and Step 3 takes 2.

    Together, we need at most $5 + 1 + 2 = \boxed{8}$ operations.

    Now we find an n matching this bound. I claim $n = 83$ works. I'll keep track of all possible values that could be obtained at each step, and verify that there's no way to reach 1 in less than 8 moves:

    1. 82
    2. 41, 81
    3. 40, 80
    4. 5, 39, 79
    5. 4, 13, 38, 78
    6. 3, 12, 19, 37, 39, 77
    7. 2, 3, 11, 13, 18, 36, 38, 76.

21. Let ω_1 and ω_2 be circles with centers O_1 and O_2, and radii 2 and 3, respectively. Let ω_3 be a third circle centered at O_3, intersecting ω_1 at point X, ω_2 at point Y, such that $O_1X \perp XO_3$ and $O_2Y \perp YO_3$. Additionally, ω_3 intersects O_1O_2 at distinct points M and N, and $O_1O_2 = 7$. Over all possible circles ω_3, compute the maximum possible value of MN.

    Answer: $\frac{24\sqrt{2}}{7}$

    Solution: Since O_1X and O_2Y are tangents to circles ω_3, the power of points O_1 and O_2 are simply the square of the respective radii $2^2 = 4$ and $3^2 = 9$. This is constant with respect to Ω_3, and gives us the system of equations:

$$\begin{cases} O_1M + MN + O_2N = 7 \\ \text{Pow}_{\omega_3}(O_1) = O_1M(O_1M + MN) = 4 \\ \text{Pow}_{\omega_3}(O_2) = O_2N(O_2N + MN) = 9. \end{cases}$$

Let $O_1M = x$, $O_2N = y$, and $MN = 7 - x - y$. We have:

$$\begin{cases} x(7-y) = 7x - xy = 4 \\ y(7-x) = 7y - xy = 9. \end{cases}$$

Subtracting the first equation from the second, we have

$$y - x = \frac{5}{7} \implies y = x + \frac{5}{7}.$$

Substituting this into the first equation:

$$\begin{aligned} 7x - x\left(x + \frac{5}{7}\right) &= 4 \\ 49x - 7x^2 - 5x &= 28 \\ 7x^2 - 44x + 28 &= 0 \\ x &= \frac{22}{7} \pm \frac{12\sqrt{2}}{7}. \end{aligned}$$

Thus,

$$\begin{aligned} MN &= 7 - x - y \\ &= 7 - x - \left(x + \frac{5}{7}\right) \\ &= 7 - 2x - \frac{5}{7} \\ &= \frac{44}{7} - 2\left(\frac{22}{7} \pm \frac{12\sqrt{2}}{7}\right) \\ &= \mp\frac{24\sqrt{2}}{7}. \end{aligned}$$

It is clear that this is only positive by choosing $x = \frac{22}{7} - \frac{12\sqrt{2}}{7}$. In this case, $MN = \boxed{\frac{24\sqrt{2}}{7}}$.

22. Call an ordered quintuple (p, q, r, s, t) of integers *good* if $-3 \leq p, q, r, s, t \leq 3$, and there exists a nonzero solution $(a, b, c, d, e) \neq (0, 0, 0, 0, 0)$ to the following system of equations:

$$\begin{cases} a + bp + cp^2 + dp^3 + ep^5 = 0 \\ a + bq + cq^2 + dq^3 + eq^5 = 0 \\ a + br + cr^2 + dr^3 + er^5 = 0 \\ a + bs + cs^2 + ds^3 + es^5 = 0 \\ a + bt + ct^2 + dt^3 + et^5 = 0. \end{cases}$$

Here, a, b, c, d, e are real numbers. Compute the number of *good* quintuples.

Answer: 14647

Solution: First of all, note that if any two of p, q, r, s and t are equal, then two of the equations are exactly equal. We can remove the redundant equation, and obtain a system of 4 equations and 5 variables. This has to have a non zero solution. Since there are $3-(-3)+1=7$ numbers between -3 and 3, the number of possible ordered integers tuples in this case is $7^5 - (7)(6)(5)(4)(3) = 14287$.

Otherwise, assume that t_i are all distinct. Let $P(x) = a + bx + cx^2 + dx^3 + ex^5$. The equations are equivalent to $P(p) = P(q) = P(r) = P(s) = P(t) = 0$. Since $P(x)$ is of at most degree 5, and p, q, r, s, t are all distinct, $P(x)$ must have roots p, q, r, s and t. However, since the coefficient of x^4 in $P(x)$ is 0, by Vieta's formula, $p+q+r+s+t=0$. In fact, the converse is true: if $p+q+r+s+t=0$, then we can set a, b, c, d, e to the coefficients of $P(x) = (x-p)(x-q)(x-r)(x-s)(x-t)$. Since $e = 1$ in this case, this solution is in fact nonzero. Therefore, for this case, we are simply looking for the number of ordered integer tuples (p, q, r, s, t) that sum to 0. As $(-3)+(-2)+(-1)+0+1+2+3=0$ and p to t cannot have duplicates, the number of ways to choose unordered tuples is the same as choosing the two integers not in the tuple to add up to 0. Clearly, the only possible pairs are $(-3, 3)$, $(-2, 2)$, and $(-1, 1)$, so the number of ordered tuples in this case is $3 \cdot 5! = 360$.

Adding these two cases up gives us $14287 + 360 = \boxed{14647}$.

23. Nine giraffes labeled 1 through 9 are running around a circular track, all starting at the same position called "position zero." Giraffe 1 runs 1 mile per hour. For $2 \le i \le 9$, giraffe i runs $1 + \frac{i}{10}$ miles per hour. At the start of the race, each giraffe decides to run the track clockwise or counterclockwise at random. Compute the expected value of the label of the first giraffe that giraffe 1 meets after the race begins.

Answer: $\frac{2057}{256}$

Solution: Without loss of generality, we assume that giraffe 1 runs clockwise. If all giraffes run clockwise, then giraffe 1 will first meet giraffe 9. This occurs with probability $\frac{1}{2^9}$. Otherwise, giraffe 1 will first meet the fastest giraffe that runs counterclockwise. This will be giraffe i with probability $\frac{1}{2^{10-i}}$, since all giraffes with labels higher than i must run clockwise. Thus, the expected value is

$$E = \left(\sum_{m=2}^{9} m\left(\frac{1}{2}\right)^{10-m}\right) + \frac{1}{2^8}(9).$$

The summation series is an arithmetic-geometric series. We compute:

$$S = \sum_{m=2}^{9} \frac{m}{2^{10-m}} \Longrightarrow 2S = \sum_{m=2}^{9} \frac{m}{2^{9-m}} = \sum_{m=3}^{10} \frac{m-1}{2^{10-m}}$$

$$\Longrightarrow S = 2S - S = \frac{10-1}{2^{10-10}} - \frac{2}{2^{10-2}} + \sum_{m=3}^{9} -\frac{1}{2^{10-m}}$$

$$= 9 - \frac{1}{128} - \sum_{m=1}^{7} \frac{1}{2^m} = 9 - \frac{1}{128} - \left(1 - \frac{1}{128}\right) = 8.$$

Thus, the expected value of the index of the first giraffe that giraffe 1 meets is

$$E = S + \frac{1}{2^8}(9) = 8 + \frac{9}{256} = \boxed{\frac{2057}{256}}.$$

24. Let S be the set of positive divisors of $N = 2025^{2025}$. A subset M of S is *maximally inconceivable* if it has the following properties: (1) $|M| \geq 2$, (2) for any two elements a, b in M, $15 \mid \frac{\text{lcm}(a,b)}{\gcd(a,b)}$, and (3) there does not exist an element in S not in M that can be added to M while preserving property (2). Compute the number of possible values of $\gcd(M)$ over all maximally inconceivable subsets M, where $\gcd(M)$ is the greatest common divisor of all of the elements in M.

Answer: 4051

Solution: $N = 2025^{2025} = 3^{8100}5^{4050}$. Every positive divisor of N can be expressed as $3^m 5^n$ for some integers (m, n) with $0 \leq m \leq 8100$, $0 \leq n \leq 4050$. Note that for any

$$a = 3^{m_a}5^{n_a} \quad \text{and} \quad b = 3^{m_b}5^{n_b},$$

we have:

$$\gcd(a, b) = 3^{\min(m_a, m_b)}5^{\min(n_a, n_b)}$$

and

$$\text{lcm}(a, b) = 3^{\max(m_a, m_b)}5^{\max(n_a, n_b)}.$$

Then, for $15 \mid \frac{\text{lcm}(a,b)}{\gcd(a,b)}$, we must have:

$$15 \mid \frac{3^{\max(m_a,m_b)}5^{\max(n_a,n_b)}}{3^{\min(m_a,m_b)}5^{\min(n_a,n_b)}} = 3^{\max(m_a,m_b)-\min(m_a,m_b)}5^{\max(n_a,n_b)-\min(n_a,n_b)}$$
$$\iff \max(m_a, m_b) \neq \min(m_a, m_b), \max(n_a, n_b) = \min(n_a, n_b).$$

This is true if and only if $m_a \neq m_b$ and $n_a \neq n_b$.
We can represent any number in S as a point on the xy plane by representing $k = 3^m 5^n$ as (m, n). S is reperesnted as the grid of lattice points $0 \leq m \leq 8100$ and $0 \leq n \leq 4050$. For our *maximally inconceivable* subset M, since the m-value and n-vaue cannot repeat over two elements, it is essentially choosing the maximal amount of lattice points from the grid while maintaining no two points on the same row or column. Since there are only 4051 values for n, $\mid M \mid = 4051$. As such, the n-values of M cover all numbers of $0 \leq n \leq 4050$, while the m-values of M cover 4051 numbers from $0 \leq m \leq 8100$. Since $\gcd(M)$ takes the lowest m-value and n-value of all elements in M, the possible values of $\gcd(M)$ are $(0, 0), (1, 0), (2, 0)...(8100 - 4051 + 1, 0) = (4050, 0)$. As such, the number of possible values of $\gcd(M)$ is $\boxed{4051}$.

25. Let I and O be the incenter and circumcenter of $\triangle ABC$, respectively, and let ω be its incircle. Suppose that the tangents from O to ω meet ω at E and F, and suppose EF meets BC at D. If the inradius of $\triangle ABC$ is 2 and $OI = OD = \sqrt{10}$, find BC^2. (There are two possible configurations. Either answer will be accepted.)

Answer: $36 + 16\sqrt{14}$ **or** $68 + 16\sqrt{14}$

Solution: Let I be the incenter and O the circumcenter of triangle ABC, and let the incircle ω have inradius $r = 2$. The incircle is tangent to BC at point J, and two tangents are drawn from O to ω, touching it at E and F. Let EF intersect BC at point D, and we are given that $OD = OI = \sqrt{10}$. Since OE and OF are tangents from O to ω, we use the Pythagorean theorem to find $OE = \sqrt{OI^2 - r^2} = \sqrt{6}$. Note that EF is perpendicular to OI and lies tangent to the incircle at E and F, which are equidistant from the foot G of the perpendicular from I to EF. Thus G is the midpoint of EF, and we find $IG = 2$, $OI = \sqrt{10}$, so $OG = \sqrt{OI^2 - IG^2} = \sqrt{6}$. Triangle OGI is right, and since $GF = IG \cdot \frac{OE}{OI} = 2 \cdot \frac{\sqrt{6}}{\sqrt{10}} = \frac{2\sqrt{6}}{\sqrt{10}}$, we get $EF = 2GF = \frac{4\sqrt{6}}{\sqrt{10}}$. From the same triangle OGD, where GD lies along EF, we have $OD = \sqrt{10}$ and $OG = \sqrt{6}$, so $GD = \sqrt{OD^2 - OG^2} = \sqrt{10 - 6} = \frac{8}{\sqrt{10}}$. Then $DF = GD - GF = \frac{8 - 2\sqrt{6}}{\sqrt{10}}$, $DE = GD + GF = \frac{8 + 2\sqrt{6}}{\sqrt{10}}$, and thus $DF \cdot DE = \frac{(8-2\sqrt{6})(8+2\sqrt{6})}{10} = \frac{64 - 24}{10} = 4$, so $DJ = 2$.

Let M be the midpoint of BC, and suppose that the triangle is configured such that $\angle OID = \angle ODI = \alpha$. In triangle IDG, we have $IG = 2$, $GD = \frac{8}{\sqrt{10}}$, and $ID = \sqrt{(GD)^2 + (IG)^2} = \sqrt{\frac{64}{10} + 4} = 2\sqrt{2}$. Then $\cos\alpha = \frac{IG}{ID} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$, $\sin\alpha = \frac{GD}{ID} = \frac{8/\sqrt{10}}{2\sqrt{2}} = \frac{4}{\sqrt{5}}$, and $\sin\angle ODM = \sin(135^\circ - \alpha) = \sin 135^\circ \cos\alpha - \cos 135^\circ \sin\alpha = \frac{\sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2} \cdot \frac{4}{\sqrt{5}} = \frac{1}{2} + \frac{2\sqrt{2}}{\sqrt{5}} = \frac{3\sqrt{2}}{2\sqrt{5}}$, so $OM = OD \cdot \sin\angle ODM = \sqrt{10} \cdot \frac{3\sqrt{2}}{2\sqrt{5}} = 3$. Now using the identity $OI^2 = R(R - 2r)$, we plug in $OI = \sqrt{10}$, $r = 2$, and get $10 = R(R - 4)$, so $R = 2 + \sqrt{14}$ (since $R > r$). Then $MC = \sqrt{R^2 - OM^2} = \sqrt{(2 + \sqrt{14})^2 - 9} = \sqrt{9 + 4\sqrt{14}}$, hence $BC = 2 \cdot \sqrt{9 + 4\sqrt{14}}$, and finally $BC^2 = 4(9 + 4\sqrt{14}) = \boxed{36 + 16\sqrt{14}}$.

But note that D could also lie on the opposite side of the tangency point J along BC, reversing the orientation of certain segments. In this mirrored configuration, GD still has magnitude $\frac{8}{\sqrt{10}}$, but now the angle $\angle ODM$ opens in the opposite direction, increasing the horizontal component of OM. Recomputing, we now have $\angle ODM = \sin^{-1}\left(\frac{5\sqrt{2}}{2\sqrt{5}}\right)$ instead, yielding $OM = 5$. With the same circumradius $R = 2 + \sqrt{14}$, we now find $MC = \sqrt{R^2 - 25} = \sqrt{(2 + \sqrt{14})^2 - 25} = \sqrt{19 + 4\sqrt{14}}$, so $BC = 2 \cdot \sqrt{19 + 4\sqrt{14}}$, and squaring gives $BC^2 = 4(19 + 4\sqrt{14}) = \boxed{68 + 16\sqrt{14}}$. Both values are thus valid depending on which direction point D lies along line BC, relative to the incircle's tangency point.

26. Call a permutation σ of $\{1, 2, ..., 9\}$ *almost-convex* if there exists at most one triple $a, b, c \in \{1, 2, ..., 9\}$ such that $a < b < c$, and

$$\sigma(b) > \frac{\sigma(c) - \sigma(a)}{c - a}(b - a) + \sigma(a).$$

Compute the number of almost-convex permutations of $\{1, 2, ..., 9\}$.

Answer: 10

Solution: In general, $\sigma(b) > \frac{\sigma(c) - \sigma(a)}{c - a}(b - a) + \sigma(a)$ if and only if $(b, \sigma(b))$ lies below the line above $(a, \sigma(a))$ and $(c, \sigma(c))$. This means if we translate all of the points, or reflect them over a vertical line, then it won't effect whether the corresponding permutation is almost-convex.

Lemma 1: If σ is an almost-convex permutation of $\{1, 2, ..., n\}$ where $n \geq 4$, then $\sigma^{-1}(n) = 1$ or $\sigma^{-1}(n) = n$. Suppose otherwise. WLOG, $\sigma^{-1}(n) > 2$ (otherwise, consider almost-convex permutation, $\sigma'(i) = \sigma(n + 1 - i)$). $(\sigma^{-1}(n), n)$ lies above the line through $(1, \sigma(1))$ and $(n, \sigma(n)$, as well as the line through $(2, \sigma(2))$ and $(n, \sigma(n))$, making σ not almost-convex.

Lemma 2: If σ is an almost-convex permutation of $\{1, 2, ..., n\}$ where $n \geq 4$, with $\sigma(1) = 1$ and $\sigma(2) = 2$, then $\sigma(i) = i$ for all $i \in \{1, 2, ..., n\}$. By considering triples $(1, 2, i)$, we see that $(i, \sigma(i))$ lies on or above the line $y = x$ for all but one i. If $\sigma(n) = n$, then all $(i, \sigma(i))$ are on or below the line $y = x$ by considering triples $(1, i, n)$ and $(2, i, n)$. It follows that $\sigma(i) = i$ for all i, since σ is a permutation with $\sigma(i) \leq i$. If $\sigma(n) \neq n$, then $\sigma(n) < n$, so $(2, \sigma(2))$ is above the line through $(1, \sigma(1))$ and $(n, \sigma(n))$. Also, $\sigma^{-1}(n) < n$, so $(\sigma^{-1}(n), n)$ is above the line through $(1, \sigma(1))$ and $(n, \sigma(n))$, making σ not almost-convex.

Now, we count almost-convex permutations, σ, of $\{1, 2, ..., 9\}$. By lemma 1, and considering reflections over $x = 5$, we may count σ with $\sigma(9) = 9$ and multiply that result by 2. $\sigma^{-1}(8)$ is 1 or 8 by lemma 1. If $\sigma(8) = 8$, then $\sigma(i) = i$ by lemma 2. This is an almost-convex permutation. Next, consider $\sigma(1) = 8$. By lemma 1, $\sigma^{-1}(7)$ is 2 or 8. If $\sigma(2) = 7$, then $\sigma(i) = 9 - i$ for $1 \leq i \leq 8$ and $\sigma(9) = 9$. This is an almost-convex permutation. Next, consider $\sigma(8) = 7$. By lemma 1, $\sigma^{-1}(6)$ is 2 or 7. If $\sigma(7) = 6$, then by lemma 2, $\sigma(i) = i - 1$ for $2 \leq i \leq 8$, $\sigma(1) = 8$ and $\sigma(9) = 9$. This is an almost-convex permutation. Next, consider $\sigma(2) = 6$. By lemma 2, $\sigma^{-1}(5)$ is 3 or 7. If $\sigma^{-1}(5) = 3$, then by lemma 2, $\sigma = (8, 6, 5, 4, 3, 2, 1, 7, 9)$. This is not almost-convex by considering triples $(6, 8, 9)$ and $(7, 8, 9)$. If $\sigma(7) = 5$, then $\sigma^{-1}(4)$ is 3 or 6 by lemma 1. If $\sigma(6) = 4$, then $\sigma = (8, 6, 1, 2, 3, 4, 5, 7, 9)$ by lemma 2. This is almost-convex. Finally, consider $\sigma(3) = 4$.

By considering triples $(6, 7, 8)$ and $(6, 7, 9)$, $\sigma(6) \geq 3$. σ is a permutation, so $\sigma(6) = 3$. By considering triples $(1, 3, 4)$ and $(1, 2, 4)$, $\sigma(4) \geq 2$. σ is a permutation, so $\sigma(4) = 2$ and $\sigma(5) = 1$. We verify that $\sigma = (8, 6, 4, 2, 1, 3, 5, 7, 9)$ is almost-convex, for a total of $5 \cdot 2 = \boxed{10}$ almost-convex permutations.

27. We define the *m-expansion* of a multiset of numbers (a multiset is a set where duplicates are allowed) as the maximum sum of squares among all of the subsets of size m (formally, the expansion is $\max_{S:\, |S| = m} \sum_{i \in S} a_i^2$).

Suppose that $a_1, a_2, ..., a_{170}$ are real numbers such that $\sum_{i=1}^{170} a_i^2 = 170$ and $\sum_{i=1}^{170} a_i^4 = 510$. Compute the smallest possible 85-expansion of any set $a_1, a_2, ..., a_{170}$ satisfying these properties.

Answer: $85 + \frac{85\sqrt{2}}{13}$

Solution: Let $n = 85$. Suppose we have ordered $a_1^2 \leq a_2^2 \leq ... \leq a_{2n}^2$. We claim that the optimal arrangement must satisfy $a_1 = a_2 = ... = a_m$ and $a_{m+1} = a_{n+2} = ... = a_{2n}$ for some m. First we show a slightly looser statement: the optimal arrangement only takes values in three sets, one of which is 0 (this set can have size 0).

To do so, for a given sequence $(a_1, a_2, ..., a_{2n})$ we will group them into *bundles* $(A_1, A_2, ..., A_k)$ by equal value. We claim that we can do a sequence of steps which do not increase the objective value such that each step, either the number of bundles decreases, or the number of a_i which are 0 increases and the number of bundles doesn't change.

To do so, take three bundles A, B, C with values $0 < v_a < v_b < v_c$ and sizes k_a, k_b, k_c respectively (if three such bundles do not exist, we are done). Let's suppose that for these bundles, ℓ_a, ℓ_b, ℓ_c values are in the top n (note that $\ell_i = k_i$ or 0 unless one of these bundles crosses the top n boundary, and then we take ℓ_i to be the number of a_j within it which are above the boundary). Then, our goal is to decrease $\ell_a v_a + \ell_b v_b + \ell_c v_c$ while keeping $k_a v_a + k_b v_b + k_c v_c = X$ and $k_a v_a^2 + k_b v_b^2 + k_c v_c^2 = Y$ constant. Note that by shifting the values of v_a, v_b, v_c the order of a_i's doesn't change, provided that no bundles change order. This also means that the top n a_i's remain constant.

We split into a few cases.

- Suppose that $\ell_b = k_b$. Then all of bundles B and C must be in the top n entries. In this scenario, our minimization is equivalent to maximizing $(k_a - \ell_a) \cdot v_a$, or equivalently increasing v_a.
- Suppose that $0 < \ell_b < k_b$. Then all of bundle C must be in the top n entries. In this case, we wish to minimize $k_c v_c + \ell_b v_b$.
- Suppose that $\ell_a = \ell_b = 0$. Then, we wish to minimize $\ell_c \cdot v_c$, or equivalently wish to decrease v_c.

For the first and last case, the unique (local) minimizer is achieved at $v_a = v_b$ or $v_b = v_c$, respectively.

For the second case, consider changing $v_{(i)}{}' = v_i + \varepsilon_i$. For a parameter $0 < s < \frac{1}{2}$, take $\varepsilon_a = -\frac{k_b - \ell_b s}{k_a}\varepsilon_b$ and $\varepsilon_c = -\frac{\ell_b s}{k_c}\varepsilon_b$. Then, $k_c\varepsilon_c + s\ell_b\varepsilon_b = 0$ and so $k_c\varepsilon_c + \ell_b\varepsilon_b \leq 0$ if $\varepsilon_b \leq 0$. Furthermore, we may compute that

$$\varepsilon_b = \frac{2k_a k_c(k_b(v_a - v_b) + \ell_b s(v_c - v_a))}{k_b(k_a + k_b)k_c - 2k_b k_c \ell_b s + (k_a + k_c)\ell_b^2 s^2}.$$

The denominator is positive as $k_b^2 k_c > 2k_b k_c \ell_b s$. Furthermore, if $s \leq \frac{k_b(v_b - v_a)}{\ell_b(v_c - v_a)}$, then $\varepsilon_b \leq 0$. Therefore, since such a parameter setting exists, we can take this $s, \varepsilon_a, \varepsilon_b, \varepsilon_c$. We see that $\varepsilon_a > 0, \varepsilon_b < 0, \varepsilon_c > 0$. Therefore, applying this step over and over leads us to having $v_a = v_b$.

Then, moving in the appropriate direction (for example, increasing v_a while decreasing v_b and modifying v_c appropriately, which is feasible by the continuity of polynomial solutions) can never increase the objective value until we hit our stopping condition: either one of the bundles hits 0 or another bundle.

We claim that it is not optimal to have any 0 bundle. Indeed, doing so immediately increases the average entry and so the expansion must increase.

From this, we can conclude that $2n = m \cdot a_1^2 + (2n - m)a_{2n}^2$ and $6n = m \cdot a_1^4 + (2n - m) \cdot a_{2n}^4$. Let $x = a_1^2 - 1$ and $y = a_{2n}^2 - 1$. Then, $m \cdot x + (2n - m) \cdot y = 0$ and $6n = m \cdot (x + 1)^2 + (2n - m) \cdot (y + 1)^2$. Substituting $x = \frac{m - 2n}{m} \cdot y$ into the latter equation yields

$$\begin{aligned} 6n &= m \cdot \left(\frac{m - 2n}{m} \cdot y + 1\right)^2 + (2n - m) \cdot (y + 1)^2 \\ &= \frac{1}{m} \cdot ((m - 2n)^2 y^2 + 2m(m - 2n)y + m^2) + (2n - m)(y^2 + 2y + 1) \\ &= y^2 \cdot (2n - m)\left(\frac{2n - m}{m} + 1\right) + 2n \\ &= y^2 \cdot (2n - m) \cdot \frac{2n}{m} + 2n. \end{aligned}$$

Dividing by $2n$ gives $y^2 \cdot \frac{2n - m}{m} = 2$, so $y = \sqrt{\frac{2m}{2n - m}}$ and $x = -\sqrt{\frac{2(2n - m)}{m}}$. Recall that $x + 1 \geq 0$, so $m \geq 2(2n - m)$; or, in other words, $m \geq \frac{4}{3}n$.

In this case, the expansion is $2n - n \cdot (x + 1)$ so we want to maximize x which can be achieved when $m = 2n - 1$ and thus $x = -\sqrt{\frac{2}{2n - 1}}$. Then, the answer is $n + n\sqrt{\frac{2}{2n - 1}} = \boxed{85 + \frac{85\sqrt{2}}{13}}$. Note that in this case, we have $a_1^2 = a_2^2 = \ldots = a_{2n-1}^2 = 1 - \frac{\sqrt{2}}{13}$ and $a_{2n}^2 = 1 + 13\sqrt{2}$.

28. A function $f : \{1, 2, ..., 100\} \to \{1, 2, ..., 100\}$ is constructed randomly (that is, each $f(i)$ for $1 \le i \le 100$ is selected from $\{1, 2, ..., 100\}$, uniformly and at random). For $2 \le k \le 100$, a length k *cycle* is a sequence of numbers $n_1, n_2, ..., n_k$ such that $f(n_i) = n_{i+1}$ for all $1 \le i \le k - 1$ and $f(n_k) = n_1$. A length 1 *cycle* is a number n satisfying $f(n) = n$. Note that a number is part of at most one cycle. Moreover, a number can be part of none: for example, if $f(1) = 2$ and $f(2) = 2$, then 1 is part of no cycles. Compute the expected sum of squares of the cycle lengths of f (including multiplicity, and each cycle of f is counted exactly once).

Answer: 100

Solution: Let C_i be the length of the cycle that i participates in (which could be 0). The sum of squares of cycle lengths is equal to $\sum_{i=1}^{100} C_i$. By symmetry and linearity of expectation,

$$\mathbb{E}\left[\sum_{i=1}^{100} C_i\right] = 100\mathbb{E}[C_1].$$

Let $k \in \{1, ..., 100\}$ be minimal such that $f^k(1) = f^\ell(1)$ for some $\ell < k$. This is well defined because $1, f^1(1), f^2(1), ..., f^{100}(k)$ has to repeat by the pigeonhole principle. Let p_j be the probability that $k = j$. 1 participates in a cycle of size j if and only if $k = j$ and $f^j(1) = 1$. Given $k = j$, we know that $f^j(1) \in \{1, f^1(1), f^2(1), ..., f^{j-1}(1)\}$, and that $\left|\{1, f^1(1), f^2(1), ..., f^{j-1}(1)\}\right| = j$. f is chosen at random, so $f^j(1) = 1$ with probability $\frac{1}{j}$, given that $k = j$. Therefore, 1 participates in a cycle of length j with probability $\frac{p_j}{j}$, so the expected value of C_1 is

$$\sum_{j=1}^{100} j \cdot \frac{p_j}{j} = \sum_{j=1}^{100} p_j = 1.$$

It follows that the final answer is $100 \cdot 1 = \boxed{100}$.

29. Welcome to the **USAYNO** (USA Yes/No Olympiad), a contest where every question has a yes/no answer! Each of the next four problems consists of 5 yes/no questions. For each of these, if you answer n questions and get them all correct, you will receive $n(n-1)$ points. If *any* of your answers for a given problem are incorrect, you will receive 0 points.

Submit your answer as a string of 5 letters, with $\boldsymbol{Y}$ representing Yes, $\boldsymbol{N}$ representing No, and $\boldsymbol{B}$ representing Blank (no submission). An example string is **BYBNY**.

1. An *algebraic number* is any real number r which is the root of some polynomial with integer coefficients. A *bi-algebraic number* is any real number r which is a root of some polynomial with coefficients being algebraic numbers. Are there numbers which are bi-algebraic but not algebraic?
2. Call a function k-periodic if $f(x) = f(x + k)$ for all real numbers x. Does there exist a series of functions $f_1, f_2, ...$ each with integer period such that $\sum_{i=1}^{\infty} f(x) = x$ for all *irrational* $x > 0$?
3. Does there exist a complex number z such that $|\operatorname{Im}(z^n)|$ is strictly increasing with n?
4. Let A be the area bounded between the x-axis, y-axis, $y = e^{\cos x^2}$, and $x = \sqrt{\frac{17\pi}{4}}$; and let B be the area bounded between the x-axis, y-axis, $y = e^{\sin x^2}$, and $x = \sqrt{\frac{17\pi}{4}}$. Is $A > B$?
5. Do there exist three positive real numbers α, β, γ such that each positive integer can be represented as exactly one of $\lfloor k\alpha \rfloor, \lfloor k\beta \rfloor, \lfloor k\gamma \rfloor$ for some positive integer k?

Answer: NYNYN

Solution: a. No. The two are equivalent (this is a consequence of the fact that algebraic numbers are algebraically closed); see for example `https : //math.stackexchange.com/questions/1430084/can-every-polynomial-with-algebraic-coefficients`

b. Yes. The irrational restriction makes this easier. Our first function will just be x in the interval $[0, 1)$ (a zigzag), and will be the only function with positive slope. The idea for all the other functions is to make sure that the sum is correct at integers'' (really, at points infinitesimally larger than the integers themselves). In particular, the i'th line (starting at $i = 2$) will be i-periodic and have value 0 from 0 through $i - 1$ and then $\varphi(i)$ from $i - 1$ to i. Then, the functions which are activated'' on an interval from $n - 1$ through n sum to $\sum_{d|n,d\neq 1} \varphi(d) = n - 1$ as desired.

c. No. A proof can be found in `https : //www.sciencedirect.com/science/article/pii/S0022314X99923946`. On an intuitive level, this should make sense: if $z = re^{i\theta}$, then $\text{Im}(z^n) = r^n \sin(n\theta)$ and we should expect (probabilistically) that $n\theta$ gets incredibly close to a multiple of π, enough to overwhelm a multiplication by r for any finite r.

d. Yes. The key idea is that $\cos(0) = 1$ and $\sin(0) = 0$, so A starts out a bit larger than B. While the two integrals cross at near $x = 2$, A eventually exceeds B. We can verify this with a computer algebra package.

e. No. Suppose that (WLOG) $\alpha \leq 1$. Then, $\lfloor k\alpha \rfloor$ iterates over all integers, and so must overlap with $\lfloor k\beta \rfloor$ for some n. Else, Uspensky's Theorem implies that this is impossible (i.e. at least one of them must be 0).

30. 1. Consider a chessboard which is an 8×8 grid of squares. A king is a chess piece which can move uniformly at random to any non-blocked position adjacent (either orthogonally or diagonally) to its current position until it returns to its original starting point. Is it possible to place the king on the chessboard and block off certain squares in such a way that the expected time until the king returns to its starting location is exactly $\frac{64}{9}$?

    2. Is it possible to color every point on the 2D plane (that is, $\mathbb{R}^2$) with finitely many colors such that any two points on the plane that are a positive integer distance apart have different colors?

    3. A class of 10 people form clubs, each consisting of 6 distinct people. Is it possible to create 120 such clubs such that no two clubs are exactly the same or have exactly 3 people in common?

    4. There are 2025 people, each of whom is given a black or red hat with equal (50-50) probability. Each person cannot see their own hat, but can see each of the other people's hats. Each person either chooses to guess their hat color or say nothing, and the group wins if at least one person guesses correctly and nobody guesses incorrectly. After the group receives their hats, aside from stating their guesses or saying nothing, the people are not allowed to communicate with each other. Is there a strategy under which the people win with probability at least $\frac{3}{4}$?

    5. You are given a set $\mathcal{S}$ of *nonempty* subsets of $\{1, 2, ..., 2025\}$ such that if $A, B \in \mathcal{S}$, then $A \cup B \in \mathcal{S}$, and there is some subset A in $\mathcal{S}$ of size at most 2. Does there always exist some $1 \leq i \leq 2025$ contained in at least half of the subsets in $\mathcal{S}$?

Answer: NNNYY

Solution: a. No. This solution requires some facts from Markov chains, and will as a result not be self contained (but we highly recommend learning about Markov chains on your own!). There are a few facts (which we do not prove) we will require from Markov chains that lead to the conclusion.

- Suppose that we let the king move around on this special chessboard for a very long (i.e. infinite) amount of time without observing it, and let π_c be the probability that it is at its starting location c after this infinite run. Then, the expected number of steps for the king to return to its starting point is $\frac{1}{\pi_c}$ (this can be proved by constructing a system of equations relating all of these expected return times).
- For each square s on the chessboard, let d_s be the number of adjacent nonblocked squares. Then, $\pi_c = \frac{d_c}{\sum_{\text{squares } s} d_s}$ (this can be proved by a property known as *time-reversibility*).

Therefore, the expected time until return can be expressed as a fraction with denominator dividing the number of adjacent nonblocked squares. However, this denominator is then at most 8, so it is impossible to make the expected return time $\frac{64}{9}$.

b. No. Consider for example the set of points $S = \{(x, 0) : x \in \mathbb{Z}\}$: that is, the number line. Each point in S is at a positive integer distance from every other point in S and thus they all must have different colors, but S is infinite so this immediately requires infinitely many colors.

c. No. At first glance, this might seem possible since there are $\binom{10}{6} = 210$ distinct possible clubs. However, the Frankl-Wilson Theorem states that in fact there can be at most $\binom{10}{2} = 55$ such clubs. Without appealing to such high powered results, we can still estimate that 120 is impossible. Consider any one club, assume WLOG it consists of members $1, 2, 3, 4, 5, 6$. Then, we can remove" $\binom{6}{3} \cdot \binom{4}{3} = 80$ possible clubs: for any 3 of these members, there cannot exist a club with them and 3 of the 4 people not in this club. This already brings us down to $210 - 80 - 1 = 129$ other possible clubs (from a fairly simple constraint!). Note we subtract 1 to eliminate the first club we created. Among these possible clubs, we have 3 cases to consider. We have the sets consisting of $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8, 9, 10\}$, which I will denote as set 1 and set 2. To create new clubs, we can pick 2 from the first set and 4 from the second set, 4 from the first set and 2 from the second set, or 5 from the first set and 1 from the second set.

Case 1: There are $\binom{6}{2} \cdot \binom{4}{4} = 15$ clubs.

Case 2: There are $\binom{6}{4} \cdot \binom{4}{2} = 90$ clubs.

Case 3: There are $\binom{6}{5} \cdot \binom{4}{1} = 24$ clubs.

Since the first and third case only result in 39 total new clubs, which is less than the 119 we need, we must make at least one club from case 2, which has 4 people from set 1, and 2 people from set 2. WLOG, let's assume that club consists of the members $\{1, 2, 3, 4, 7, 8\}$. This eliminates the following 11 clubs from cases 1, 2, and 3:
$\{1, 2, 5, 6, 7, 9\}, \{1, 2, 5, 6, 7, 10\}, \{1, 2, 5, 6, 8, 10\}, \{1, 2, 5, 6, 8, 9\}, \{1, 2, 5, 6, 7, 10\}, \{1, 3, 5, 6, 7, 9\}, \{1, 3, 5, 6,$
Thus, if any club from the second case is used (which it must be), at least 11 more clubs are no longer possible, resulting in at most $129 - 1 - 11 = 117$ other clubs.

d. Yes. Let's look at the function $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2^5$ defined via $f(a, b, c) = (a, b, c, a \oplus b, a \oplus c)$ (here, $\oplus$ means addition modulo 2) and in particular let its image be $\mathcal{C} \subseteq \mathbb{Z}_2^5$. Note that if $x \in C$, flipping any single index makes x no longer in $\mathcal{C}$ (flipping one of the inputs must flip one of the latter two bits, and flipping one of the latter two immediately doesn't satisfy the requirement of

f). Finally, note that $\frac{|\mathcal{C}|}{|\mathbb{Z}_2^5|} = \frac{2^3}{2^5} = \frac{1}{4}$ (the numerator is because each element of $\mathcal{C}$ is defined by its inputs).

Now, consider the hats of just the first 5 people, assign black = 1, red = 0, and create an ordered 5-tuple of 1's and 0's using the hat colors of the first 5 people. Then, the first person will guess the hat color for themselves which makes the first 5 hat colors (viewed as an ordered tuple described prior) not an element of $\mathcal{C}$. By the above observation, it is always possible to do this as flipping the first bit will make the tuple no longer in $\mathcal{C}$. Then, the players win if and only if their hats do not actually form some $x \in \mathcal{C}$, which is exactly probability $\frac{3}{4}$.

e. Yes. Let's consider cases using the size of A. If $A = \{x\}$ (size 1), then note that at least half of the subsets must contain x (since if B does not, then $B \cup \{x\}$ does and we can thus uniquely pair up all subsets which do not contain x).

Else if $A = \{x, y\}$, suppose that x is not contained in more than half of the subsets in S. For any such subset not containing x, consider $B \cup A$ (which is not equal to B). Each such subset contains y, and by hypothesis of x not being in more than half of the subsets in S, it follows that y must be in at least half of the subsets.

31. 1. Given a square piece of paper, is it possible to (using a straightedge, compass, and by folding the piece of paper) construct three points A, B, and C such that $\frac{|AB|}{|BC|} = \sqrt[3]{2}$?
    2. Let S be the sphere in $\mathbb{R}^3$ with equation $x^2 + y^2 + z^2 = 1$. Is the surface area of the portion of S lying on or above the plane $z \geq \frac{1}{2}$ less than π?
    3. Given the plane ($\mathbb{R}^2$) initially colored white, you have two drawing implements: a pencil'', which in one move colors a unit disk (including its boundary) black, and an eraser'', which in one move colors a unit disk (including its boundary) white. With just these two implements and a possibly infinite number of moves, is it possible to make it so the only black part of the plane is a filled in regular pentagon (including its boundary) with side length $\frac{1}{5}$?
    4. Let $O = (0, 0)$, $A = (1, 0)$, and $B = (0, 1)$. For any C on OA, let D be the point on OB such that $OC = BD$ and $AC = OB$, and let S be the area swept out by CD over all choices of C. Is S larger than $1 - \frac{\pi}{4}$?
    5. Consider placing 4 distinct points in a plane such that no three of them lie on line, and the edges connecting each pair of vertices don't intersect in their interiors. Is it possible to place these points in such a way that every edge has integer length?

Answer: YNYNY

Solution: a. Yes. For example, see here: `http : //origametry.net/omfiles/geoconst.html` (How to double a cube via folding''). The construction is not too bad to do on paper.

b. No. In fact, the surface area element of the 3-d sphere is constant, so the surface area above this plane is (surprisingly) exactly π.

c. Yes. We can represent the pentagon as the set of points satisfying a set of 5 inequalities representing the sides (let us give each of these an orientation such that the pentagon lies to the left of each of them). We may begin by choosing one of these inequalities and coloring all points on the pentagon side black (using an infinite number of moves). Then, for each of the other inequality constraints, we can erase all points on the right of it. Note that the side length does not matter; only convexity. For more details or if you're interested in the formalization of drawing, feel free to take a look at `https : //arxiv.org/abs/2004.01049`.

d. No. We can show that the equation of the curve defining S is $\sqrt{x}+\sqrt{y}=1$, so the area under the curve is $\int_0^1 (1-\sqrt{x})^2\,\mathrm{d}x = \frac{1}{6}$.

e. Yes. Note that this placing must look like a triangle ABC with a point P inside it, connected to the three corners. One way we can think to do this is to find two Pythagorean triples (a,b,c) and (a,d,e) (with $e>c$). Then, we can construct the desired triangle with side lengths $AB = 2a, AC = e, BC = e$ and distances from P being $PA = c, PB = c, PC = d - b$. One such possibility is then $(a,b,c) = (8,6,10)$ and $(a,d,e) = (8,15,17)$. Note, this problem can also be viewed as an embedding of a tetrahedron into the plane.

32. 1. Are n^3+12 and $(n+1)^3+12$ relatively prime for all integers n?

2. Define the 2-parse of a positive integer as a way to split its decimal digits into contiguous segments where each segment represents a power of 2. For example, the integer 23221 has the 2-parse $[2, 32, 2, 1]$, because 2, 32, 2, and 1 are all powers of 2. Does there exist a positive integer with more than one distinct 2-parse?

3. Does there exist a prime with more than two distinct digits such that any permutation of its digits yields a prime? Disregarding the distinctness constraint, a two-digit example is 13, since 31 is also prime.

4. Every positive integer can be written as the sum of at most 9 positive integer cubes. Let $f(n)$ be the minimum number of positive integer cubes whose sum is n. Among the integers $1 \le n \le 2025$, are there more values of n for which $f(n) = 3$ than for which $f(n) = 6$?

5. Let a and b be distinct positive integers whose digits contain no zeroes. Define the infinite string s formed by the concatenation $baaaa\cdots$ and interpret each prefix of s as a base-10 integer. A prime p is good'' if p divides at least one prefix of s. Does there exist a, b such that infinitely many primes are good and infinitely many primes are not good?

Answer: **NYNNY**

Solution: a. No. A counterexample is $n = 1926$, where the gcd is $3889 > 1$. In general, we may compute that the resultant (`https://en.wikipedia.org/wiki/Resultant`) of these two polynomials is 3889. Examining this in $\mathbb{F}_{3889}$, it follows that these two polynomials share a linear factor there and so there must be an n for which they are not relatively prime.

b. Yes. For example, 164 can be parsed as either $[1, 64]$ or $[16, 4]$ (yes, sometimes there is that small of a counterexample!). In fact, even 128 is ambiguous.

c. No. See `https://oeis.org/A003459/a003459.pdf` for more details (the idea is we can then always construct a number divisible by 7 when there are sufficiently many digits, and for smaller numbers of digits we can just check directly).

d. No. Let $F(k)$ denote the number of integers $1 \le n \le 2025$ with $f(n) = k$. One intuition is that we would expect

$$F(9) < F(1) < F(8) < F(2) < F(7) < F(3) < F(6) < F(4) < F(5).$$

A reason for this is that the distribution of f should be bottom-heavy'', but 6 is close enough to the center (5) that it should still overwhelm 3. Indeed, $F(6) = 401$ while $F(3) = 239$.

e. Yes. Consider $b = 2$ and $a = 1$ (really, pretty much any choice of a and b works but proving it is a bit difficult). Then, via some rearrangement, we see that a large enough prime p is good iff there exists some n with $10^n \equiv 19 \bmod p$.

For the forward direction, suppose that there are finitely many good primes $p_1, p_2, ..., p_k$ (we can check that there are at least two: $p_4 = 23$ is a good prime and so is $p_1 = 3$). Then, let m be their product. Note that $10^{\varphi(m)+1} - 19 \equiv -9 \bmod m$ and $10^{\varphi(m)+2} - 19 \equiv 81 \bmod m$, so consider $n = \varphi(m) + 1$. We claim that at least one of $10^n - 19$ and $10^{n+1} - 19$ has a prime divisor $p > 3$. Note that both cannot be powers of three for $n \geq \varphi(3 \cdot 23) = 44$ (we would require $10 \cdot 10^n - 19 = 9(10^n - 19)$ which has no positive solutions). Then, this prime divisor p does not divide m, as it does not divide 9 or 81. Therefore, there must be an infinite number of good primes.

In the backward direction, by quadratic reciprocity we know that 10 is a quadratic residue for primes p with $p \equiv 1 \bmod 40$, and 19 not a quadratic residue for primes p with $p \equiv 5 \bmod 76$. Therefore, by the Chinese Remainder Theorem, we can determine that both of these facts hold for any prime $p \equiv 81 \bmod 760$. Since by Dirichlet's Theorem there are infinitely many such primes, we have our conclusion.