1.
$$\int \frac{1}{x+\sqrt{x}}\,dx$$

Answer: $2\ln|\ \sqrt{x}+1\ |+C$

Solution: Take a u-substitution $x = u^2$ to get:

$$\int \frac{2u}{u^2+u}\,du = \int \frac{2}{u+1}\,du = 2\ln|\ u+1\ |+C = 2\ln|\ \sqrt{x}+1\ |+C$$

2.
$$\int_0^\infty \frac{4x}{1+x^4}\,dx$$

Answer: π

Solution: Do a u-sub with $u = x^2$.

3.
$$\int_1^\infty \frac{1}{x^2\sqrt{x^2-1}}\,dx$$

Answer: 1

Solution: trig sub

4.
$$\int_1^\infty \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\,dx$$

Answer: $1-\ln(2)$**.**

Solution: Integration by parts by deriving $\ln(1+\frac{1}{x})$ as a whole. Then

$$\left[x\ln\left(1+\frac{1}{x}\right)\right]_1^\infty = 1-\ln(2)$$

5.
$$\int \frac{\sin^4\theta}{\cos\theta}\,d\theta$$

Answer: $-\frac{\sin^3\theta}{3} - \sin\theta + \ln|\sec\theta+\tan\theta\ |+C$

Solution: Simplify using trig identities:

$$\begin{aligned}\int \frac{(1-\cos^2\theta)^2}{\cos\theta}\,d\theta &= \int \cos^3\theta - 2\cos\theta + \sec\theta\,d\theta \\ &= \int \cos\theta(1-\sin^2\theta) - 2\cos\theta + \sec\theta\,d\theta \\ &= \int -\sin^2\theta\cos\theta - \cos\theta + \sec\theta\,d\theta \\ &= -\frac{\sin^3\theta}{3} - \sin\theta + \ln|\sec\theta+\tan\theta\ |+C\end{aligned}$$

6.
$$\int_0^\infty 2 - \frac{x}{\sqrt{x^2+\pi^2}} - \frac{x}{\sqrt{x^2+e^2}}\,dx$$

Answer: $e+\pi$**.**

Solution: Basic integration gives

$$= \left[2x - \sqrt{x^2+\pi^2} - \sqrt{x^2+e^2}\right]_0^\infty$$

Note that for any real value a:

$$\lim_{x\to\infty}(x - \sqrt{x^2+a^2}) = 0.$$

7.

$$\int_{-1}^{8} \sqrt{x}\sin(\tan^{-1}(\sqrt{x}))\,dx$$

Answer: 12

Solution: the integrand becomes $\frac{x}{\sqrt{x+1}}$

8.

$$\int \frac{\sin 2x}{1+\cos x}\,dx$$

Answer: $2(\ln|\csc x + \cot x\ | - \ln|\csc x\ | - \cos x) + C$

Solution: Use trig rules and simplify/integrate.

9.

$$\int \frac{x+1}{x^2+x\ln x}\,dx$$

Answer: $\ln(x+\ln x)$

Solution: u-sub

10.

$$\int_0^{\frac{3}{2}} \sqrt{1+\sqrt{x^2+\sqrt{x^6+\sqrt{x^{14}+\sqrt{x^{30}+\cdots}}}}}\,dx$$

Answer: $\frac{3}{2}+\ln(2)$.

Solution: Notice that $y = \sqrt{1+xy}$. Then $\frac{y^2-1}{y} = x$. By inverse sum formula,

$$\int_0^{3/2} f(x)dx + \int_1^2 f^{-1}(x)dx = 3$$

Hence,

$$I = 3 - \int_1^2 x - \frac{1}{x}dx = \boxed{\frac{3}{2}+\ln(2)}$$

11.

$$\int_0^{3/2} \frac{1}{x+\frac{1}{x+\dots}}\,dx$$

Answer: $16\ln 2 + 3$

Solution: Let $u = \frac{1}{x+\frac{1}{x+\dots}}$, solve for u and x in terms of the other, and do a u-sub.

12.

$$\int_0^{\pi} \frac{6+6\cos^2 x}{1+3\sin^2 x}\,dx$$

Answer: 5π

Solution: Trig

13.
$$\int_0^1 \sqrt{x+x^3}+\sqrt{x+\sqrt[3]{x}}\,dx$$

Answer: $\boxed{\frac{4\sqrt{2}}{3}}$.

Solution: On the second term, let $u^3 = x$. Then we have

$$\int_0^1 \sqrt{x+\sqrt[3]{x}}dx = \int_0^1 3u^2\sqrt{u^3+u}du.$$

Then it follows

$$\int_0^1 \sqrt{x+x^3}+\sqrt{x+\sqrt[3]{x}}dx = \int_0^1 (3x^2+1)\sqrt{x+x^3}dx = \left[\frac{2}{3}(x+x^3)^{\frac{3}{2}}\right]_0^1 = \frac{4\sqrt{2}}{3}.$$

14.
$$\int_0^{\frac{\pi}{4}} \sum_{n=2}^{\infty}\left(\frac{1}{\sqrt{n-\sec^2(x)}}-\frac{1}{\sqrt{n-\tan^2(x)}}\right)dx$$

Answer: $\boxed{\frac{\pi}{2\sqrt{2}}}$.

Solution: Using the trig-identitiy makes the sum telescope to $\frac{1}{\sqrt{1-\tan^2(x)}}$.

$$= \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{1-\tan^2(x)}} = \int_0^1 \frac{du}{(1+u^2)\sqrt{1-u^2}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin^2\theta}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{1+2\tan^2\theta}d\theta = \frac{\pi}{2\sqrt{2}}$$

15.
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4\log^2(\csc^2 x)}{\tan x}\,dx$$

Answer: $\log^2(2)$

Solution: u-sub with $u = \sin^2 x$

16.
$$\int e^{17x}(\cos^3 x - 3\sin^2 x\cos x)\,dx$$

Answer: $\frac{e^{17x}(17\cos(3x)+3\sin(3x))}{298} + C$

Solution: The trigonometric part is $\cos(3x)$, so one can use Euler's Identity and take the real part of the integral, or just do integration by parts.

17.
$$\int e^{x\tan^{-1}(x)}\frac{\tan^{-1}(x)}{\sqrt{x^2+1}}\,dx$$

Answer: $\boxed{e^{x\tan^{-1}(x)-\ln(\sqrt{x^2+1})}+C}$

Solution: Very sneaky to see, but

$$\frac{1}{\sqrt{x^2+1}} = e^{-\ln(\sqrt{x^2+1})}.$$

Then let $u = x\tan^{-1}(x) - \frac{1}{2}\ln(x^2+1)$ to easily solve the integral.

18.
$$\int_0^{\varphi} e^{x-\frac{1}{x}}(x+1)^2\,dx$$

Answer: $\boxed{\varphi^2 e}$.

Solution:

$$= \int_0^{\varphi} \left(1+\frac{1}{x^2}\right)e^{x-\frac{1}{x}} \cdot x^2 + 2x \cdot e^{x-\frac{1}{x}}\,dx$$

$$= \left[x^2 e^{x-\frac{1}{x}}\right]_0^{\varphi} = \varphi^2 e$$

19.
$$\int \frac{1}{x^5+x^3} - \frac{3\tan^{-1}(x)}{x^4}\,dx$$

Answer: $\frac{\tan^{-1}(x)}{x^3} + C$

Solution: It's best to go backwards with this one. As complicated as this looks, it's actually a quotient rule in disgused. The x^{-3} in the first term should be suggestive, along with x^2+1 term. We compute the derivative of $\frac{\tan^{-1}(x)}{x^3}$ and go backwards:

$$\begin{aligned}\frac{\frac{x^3}{x^2+1} - 3x^2\tan^{-1}(x)}{x^6} &= \frac{x^3 - 3x^2(x^2+1)\tan^{-1}(x)}{x^6(x^2+1)} \\ &= \frac{1}{x^5+x^3} - \frac{3\tan^{-1}(x)}{x^4}\end{aligned}$$

20.
$$\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt[4]{2}+\sqrt{1+\tan(x)}}\,dx$$

Answer: $\boxed{\frac{\pi}{4\sqrt[4]{2}}}$

Solution: Secretly a King's rule problem, even if it doesn't look it. Let $u = \frac{\pi}{4} - x$.

$$\begin{aligned}I &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt[4]{2}+\sqrt{1+\tan(\frac{\pi}{4}-u)}}du \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt[4]{2}+\sqrt{1+\frac{1-\tan(u)}{1+\tan(u)}}}du \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt[4]{2}+\sqrt{\frac{2}{1+\tan(u)}}}du \\ &= \int_0^{\frac{\pi}{4}} \frac{\frac{\sqrt{1+\tan(u)}}{\sqrt[4]{2}}}{\sqrt{1+\tan(u)}+\sqrt[4]{2}}du.\end{aligned}$$

Hence,

$$2I = \int_0^{\frac{\pi}{4}} \frac{1+\frac{\sqrt{1+\tan(x)}}{\sqrt[4]{2}}}{\sqrt{1+\tan(x)}+\sqrt[4]{2}}dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt[4]{2}}dx = \frac{\pi}{4\sqrt[4]{2}}$$