1.
$$\int \frac{e^{4x}}{(e^{2x}+1)^2}\,dx$$

Answer:

$$\frac{1}{2}\left(\ln|\, e^{2x}+1\,| + \frac{1}{e^{2x}+1}\right)+C$$

Solution: First, immediately apply a u-substitution $u = e^{2x}+1$ so that:

$$\int \frac{1}{2}\frac{u-1}{u^2}\,du = \int \frac{1}{2}\left(\frac{1}{u}-\frac{1}{u^2}\right)\,du = \frac{1}{2}\left(\ln|\,u\,| + \frac{1}{u}\right)$$

so with a back-subsitution, we get the desired answer.

$$\frac{1}{2}\left(\ln|\, e^{2x}+1\,| + \frac{1}{e^{2x}+1}\right)$$

2.
$$\int \sec^4 x\,dx$$

Answer: $\tan x + \frac{\tan^3 x}{3} + C$

Solution: $u = \tan x$, then power rule.

3.
$$\int_{-5}^{5} x^5 \sec x\,dx$$

Answer: 0

Solution: Obvious by symmetry.

4.
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin(3x)}{\sin(x)}\,dx$$

Answer: π

Solution: Expand using trig identities:

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin(2x)\cos(x)+\sin(x)\cos(2x)}{\sin(x)}\,dx &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\cos^2(x)+\cos(2x)\,dx \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\cos(2x)+1\,dx \\ &= \sin(2x)+x\,\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\ &= \pi\end{aligned}$$

5.
$$\int \frac{2x^2-1}{x^4-x^2-2}\,dx$$

Answer: $\tan^{-1}(x) + \frac{1}{2\sqrt{2}} \ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + C = \tan^{-1}(x) - \frac{1}{\sqrt{2}} \tanh^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

Solution: Apply partial fraction decomposition, and we get:

$$\int \frac{1}{x^2+1} + \frac{1}{2\sqrt{2}}\left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right) dx = \tan^{-1}(x) + \frac{1}{2\sqrt{2}} \ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|$$

Note that this is the same as:

$$\int \frac{1}{x^2+1} + \frac{1}{x^2-2}\, dx$$

6.
$$\int \frac{x^{11}+x^5}{\sqrt[3]{x^6+6}}\, dx$$

Answer: $\frac{1}{10}(x^6+6)^{\frac{5}{3}} - \frac{5}{4}(x^6+6)^{\frac{2}{3}} + C$

Solution: $u = x^6 + 1$, then proceed

7.
$$\int \frac{x^{14}}{x^5+1}\, dx$$

Answer: $\frac{1}{5}\left(\frac{x^{10}}{2} - x^5 - \frac{3}{2} + \ln|\ x^5+1\ |\right) + C$

Solution: Take $x^5 + 1 = u$, so we have:

$$\int \frac{1}{5}\frac{(u-1)^2}{u} = \frac{1}{5}\int u - 2 + \frac{1}{u}\, du = \frac{1}{5}\left(\frac{u^2}{2} - 2u + \ln|\ u\ |\right) + C$$

so back-substitution and simplifications yield the desired answer:

$$\frac{1}{5}\left(\frac{(x^5+1)^2}{2} - 2(x^5+1) + \ln|\ x^5+1\ |\right) + C$$

$$= \frac{1}{5}\left(\frac{x^{10}}{2} - x^5 - \frac{3}{2} + \ln|\ x^5+1\ |\right) + C$$

8.
$$\int \frac{(x-1)(x^2-1)}{(x^3-1)(x^4-1)}\, dx$$

Answer: $\frac{1}{2}\ln|\ x^2+x+1\ | + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{1}{2}\ln|\ x^2+1\ | + C$

Solution: Partial fractions, then standard integration techniques

9.
$$\int_2^3 (5-x)^{2025}(x^2-10x+24)dx$$

Answer: $\frac{3^{2028}-2^{2028}}{2028} - \frac{3^{2026}-2^{2026}}{2026}$

Solution: Apply King's Identity, then solve via power rule.

10.
$$\int_0^1 \sqrt{x\sqrt[3]{x\sqrt[4]{x\sqrt[5]{x\ldots}}}}\,dx$$

Answer: $\frac{1}{e-1}$

Solution: This is a Taylor series, which is of the form:

$$x^{e-2}$$

So integrating is basic power rule

11.
$$\int \frac{1}{1+\sec(x)}\,dx$$

Answer: $x-\csc(x)+\cot(x)+C$

Solution: Just uses properties of trigonometry

12.
$$\int e^{3x}\left(\frac{\ln^2 x}{x}+\ln^3 x\right)\,dx$$

Answer: $\frac{e^{3x}\ln^3 x}{3}+C$

Solution: U-sub $u=e^x\ln x$

13.
$$\int_0^\infty xe^x\sin(x)dx$$

Answer: $\frac{1}{2}$

Solution:

14.
$$\int_0^1 \sqrt[\varphi]{\frac{x}{x^\varphi+1}}dx$$

Answer: $\frac{2^{1-1/\varphi}-1}{\varphi-1}$**.**

Solution:

$$=\int_0^1 \frac{x^{1/\varphi}}{\sqrt[\varphi]{x^\varphi+1}}dx$$

Let $u=x^\varphi+1$, $du=\varphi x^{\varphi-1}dx$. Note that $\varphi-1=\frac{1}{\varphi}$. Then $du=\varphi x^{1/\varphi}dx$.

$$=\frac{1}{\varphi}\int_1^2 \frac{du}{u^{1/\varphi}}=\frac{1}{\varphi}\left[\frac{u^{1-1/\varphi}}{1-\frac{1}{\varphi}}\right]_1^2=\boxed{\frac{2^{1-1/\varphi}-1}{\varphi-1}}.$$

15.
$$\int_{-\infty}^{\infty}(-2x^8+x^5)e^{-x^6+x^3+7}\,dx$$

Answer: $-\frac{e^{\frac{29}{4}}\sqrt{\pi}}{3}$

Solution: Apply integration by parts with $\frac{x^3}{3}$ and the rest, then apply a u-sub $u = x^3$.

16.
$$\int_0^\infty \frac{\lfloor x \rfloor}{(x+1)^3}\, dx$$

Answer: $\frac{\pi^2}{12} - \frac{1}{2}$

Solution:

17.
$$\int_0^1 x^2 \tan^{-1}(x)\, dx$$

Answer: $\frac{1}{6}(\frac{\pi}{2} - 1 + \ln 2)$

Solution: Apply $x = \tan\theta$, then integration by parts.

18.
$$\int_{-\infty}^\infty \sin(e^x)\, dx$$

Answer: $\frac{\pi}{2}$

Solution: $u = e^x$, then note the integrand is sinx/x (should be known integral somewhat)

19.
$$\int_2^3 \ln\left(\prod_{n=1}^\infty \sum_{k=0}^\infty \frac{n^k}{x^{nk}\cdot k!}\right) dx$$

Answer: $-\frac{1}{2} - \ln(2)$.

Solution:

$$\int_2^3 \ln\left(\prod_{n=1}^\infty \sum_{k=0}^\infty \frac{(nx^{-n})^k}{k!}\right) dx = \int_2^3 \ln\left(\prod_{n=1}^\infty e^{nx^{-n}}\right) dx = \int_2^3 \sum_{n=1}^\infty \frac{n}{x^n} dx$$

$$= \int_2^3 -\frac{\frac{1}{x}}{(1-\frac{1}{x})^2} dx = \int_2^3 -\frac{x}{(x-1)^2} dx = \int_2^3 -\frac{1}{x-1} - \frac{1}{(x-1)^2} dx$$

$$= \left[\frac{1}{x-1} - \ln|\ x-1\ |\right]_2^3 = \boxed{-\frac{1}{2} - \ln(2)}$$

20.
$$\int (\cos x)^x (x \ln\cos x - x^2 \tan x + 1)\, dx$$

Answer: $x(\cos x)^x + C$

Solution: Realize its an antiderivative of the form.

21.
$$\int \sqrt{\sin^4(x) + 4\cos^2(x)} - \sqrt{\cos^4(x) + 4\sin^2(x)} dx$$

Answer: $\frac{1}{2}\sin(2x) + C$

Solution: Let's abbreviate $\sin(x)$ and $\cos(x)$.

$$= \int \sqrt{s^4+4c^2} - \sqrt{c^4+4s^2} = \int \sqrt{(1-c^2)^2+4c^2} - \sqrt{(1-s^2)^2+4s^2}$$

$$= \int \sqrt{c^4+2c^2+1} - \sqrt{s^4+2s^2+1} = \int (\cos^2(x)+1) - (\sin^2(x)+1)dx$$

$$= \int \cos(2x)dx = \boxed{\frac{\sin(2x)}{2}+C}$$

22.
$$\int_{-\pi}^{\pi} \frac{dx}{(1+e^{\cos(x)})(1+e^{x\cos(x)})}$$

Answer: $\frac{\pi}{2}$

Solution: Let $u=-x$:

$$= \frac{1}{2}\int_{-\pi}^{\pi} \frac{dx}{1+e^{\cos(x)}}$$

Let $u=\pi-x$:

$$= \int_0^{\pi} \frac{dx}{1+e^{-\cos(x)}} = \frac{1}{2}\int_0^{\pi} 1dx = \boxed{\frac{\pi}{2}}$$

This is a double abuse of King's Identity

23.
$$\int x^2 \sin^{-1}(x)\ dx$$

Answer: $\frac{x^3\sin^{-1}(x)}{3} + \frac{x^2\sqrt{1-x^2}}{9} + C$

Solution: Let $x=\sin(u)$, so we have:

$$\begin{aligned}\int u\sin^2(u)\cos(u)\ du &= \frac{u\sin^3(u)}{3} - \int \frac{\sin^3(u)}{3}\ du\\ &= \frac{u\sin^3(u)}{3} - \int \frac{(1-\cos^2(u))\sin(u)}{3}\ du\\ &= \frac{u\sin^3(u)}{3} + \cos(u) - \frac{\cos^3(u)}{9} + C\end{aligned}$$

so back-substitution yields the solution:

$$\frac{x^3\sin^{-1}(x)}{3} + \sqrt{1-x^2} - \frac{(1-x^2)^{\frac{3}{2}}}{9} + C = \frac{x^3\sin^{-1}(x)}{3} + \frac{x^2\sqrt{1-x^2}}{9} + C$$

24.
$$\int_0^{\frac{\pi}{4}} \cos(x)\sqrt{\cos(2x)} + \sin(2x)\sqrt{\sin(4x)}\ dx$$

Answer: $\boxed{\frac{\pi}{8} + \frac{\pi}{4\sqrt{2}}}$

Solution: For first term:

$$\int_0^{\pi/4} \cos(x)\sqrt{1-2\sin^2(x)}dx = \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1-2u^2}du = \frac{\pi}{4\sqrt{2}}$$

For the second term, we use $u = \frac{\pi}{2} - x$ trick:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(x)\sqrt{\sin(2x)}dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\sin(x)+\cos(x))\sqrt{1-(\sin(x)-\cos(x))^2}dx$$

$$= \frac{1}{4}\int_{-1}^{1} \sqrt{1-u^2}du = \frac{\pi}{8}$$

25.

$$\int_0^1 \frac{dx}{(1+x^2)(1+\frac{\pi}{2}+\tan^{-1}(x)\cot^{-1}(x))}$$

Answer: $\frac{\ln(1+\frac{\pi}{2})}{2+\frac{\pi}{2}}$.

Solution: Note that $\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x)$. Then,

$$= \int_0^1 \frac{dx}{(1+x^2)(1+\tan^{-1}(x))(1+\cot^{-1}(x))}$$

$$= \frac{1}{2+\frac{\pi}{2}}\int_0^1 \frac{2+\tan^{-1}(x)+\cot^{-1}(x)}{(1+x^2)(1+\tan^{-1}(x))(1+\cot^{-1}(x))}dx$$

$$= \frac{1}{2+\frac{\pi}{2}}\int_0^1 \frac{1}{(1+x^2)(1+\cot^{-1}(x))} + \frac{1}{(1+x^2)(1+\tan^{-1}(x))}dx$$

$$= \frac{1}{2+\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} \frac{1}{1+\frac{\pi}{2}-u} + \frac{1}{1+u}du = \boxed{\frac{\ln(1+\frac{\pi}{2})}{2+\frac{\pi}{2}}}$$

26.

$$\int_1^\infty \left\lfloor \frac{2}{\{x\}} \right\rfloor^{-\lfloor x \rfloor} dx$$

Answer: $\frac{1}{2}$.

Solution:

$$= \sum_{n=1}^{\infty}\int_n^{n+1} \left\lfloor \frac{2}{x-n} \right\rfloor^{-n} dx = \sum_{n=1}^{\infty}\int_0^1 \left\lfloor \frac{2}{u} \right\rfloor^{-n} du = \sum_{n=1}^{\infty}\int_2^\infty \lfloor w \rfloor^{-n} \cdot \frac{2}{w^2}dw$$

$$= \sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\int_k^{k+1} k^{-n}\frac{2}{w^2}dw = \sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n}\left[\frac{2}{k} - \frac{2}{k+1}\right]$$

$$= \sum_{k=2}^{\infty} 2\left(\frac{1}{k-1}\right)\left(\frac{1}{k} - \frac{1}{k+1}\right) = \boxed{\frac{1}{2}}$$

27.
$$\int_0^{2025} \min\left(\sqrt{\{x\}-\{x\}^2}, \left|\frac{\sqrt{3}\arcsin(\cos(\pi x))}{\pi}\right|\right) dx$$

Answer: $\frac{2025\pi}{12}$.

Solution: Understanding the graph of this function, you get a bunch of sectors of a circle with radius 1/2 and angle 60 degrees. Now you just have to count them from 0 to 2025.