Preliminaries

Time Limit: 80 minutes of work time and 5 minutes after pencils/pens down to organize pages and put them in the envelope.

Maximum Score: 111

ATTENTION: This packet only contains problems. Do not submit anything from this packet!

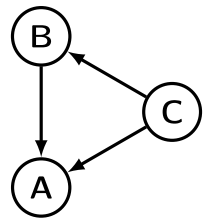
Instructions: Problems that use the words "compute", "list", "identify", or "draw" only call for an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof.

- Each team will receive exactly one packet with answer boxes (**NOT** this packet!). The boxes on the answer packet are all appropriately sized for the length of the question's answer.
- Write all answers in the corresponding provided box (the box that comes directly after the question). Do **NOT** write on the back of any pages of the answer packet. Any work written on the back of packet pages will not be graded.
- If a single problem has multiple parts, all answers must go into the same specified box. Problems with multiple boxes will be clearly specified. You may not write answers to more than one question in a box or use a different question's box to continue your answer.
- Use blank scratch paper to work out problems. You may grab more scratch paper as needed. Afterward, it is recommended to transfer your final answers into the boxes in dark pencil. Your team will not get another packet with boxes if someone used a pen.
- If your team needs more room for a problem, indicate this in the corresponding answer box **and** on the front page. Then, use one of the provided extra pages, namely pages 25 and 26, to continue your answer. Label the answer with the problem number. If you are writing multiple answers on an extra page, label each and separate them clearly using one horizontal line between problems.
- You may only use **one** side of each extra page (the side with text). If an extra page has work on two sides, neither side will be graded.
- Each team is limited to exactly **two** extra pages. Use them wisely!
- Each team will submit **all 26 pages** in the answer packet even if nothing has been written. They **must** be in the original page order. Misordered pages will not get graded and your score for those problems default to 0. Absolutely no other pages are to be submitted!

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them. The problems are ordered by content, **NOT DIFFICULTY**. It is to your advantage to attempt problems from throughout the test. While completing the round, you should not consult the internet or any materials outside of the content of this test (including results not covered in this power round). You may not use calculators.

Important: If a problem asks you to **construct** a tournament (or directed graph), you must neatly draw a directed graph with the proper vertices (labels enclosed in circles!) and **clearly** drawn directed edges (this means arrows!). If you use graphs in a proof, they **must** be in this format in order to be considered (using ellipses to denote an unknown number of edges/vertices is permitted). Make sure your graphs are large, clean, and legible.

As an example, here is the directed graph with vertices $\{A, B, C\}$ and edges $\{(B \to A), (C \to A), (C \to B)\}$.



1 Tournaments and Transitivity

Definition 1.1. A *directed graph* is an ordered pair $G = (V, E)$ where V is a set containing elements called vertices and E is a set of ordered pairs of elements of V. Each element in E is called a directed edge. If $p, q \in V$ are chosen such that there exists a directed edge $(p, q) \in E$, we denote this directed edge as $(p \to q)$.

Problem 1.1 (1pt)**.** *Construct a directed graph with vertices* $\{A, B, C, D\}$ *and edges* $\{(D \to A), (B \to C), (C \to A), (C \to D), (B \to A)\}$.

A natural way to model tournaments is to use directed graphs!

Definition 1.2. A *tournament* $T = (V, E)$ is a directed graph such that the vertices of V are called players, and there exists exactly one directed edge between every pair of players. In other words, if p and q are distinct players in T, then either $(p \to q)$ or $(q \to p)$. If $(p \to q)$, we say that p beats/wins against q.

The graph in Problem 1.1 is almost a tournament!

Problem 1.2 (1pt)**.** *Redraw the graph in problem 1.1 and modify it to make it a tournament. (There are multiple possible constructions.)*

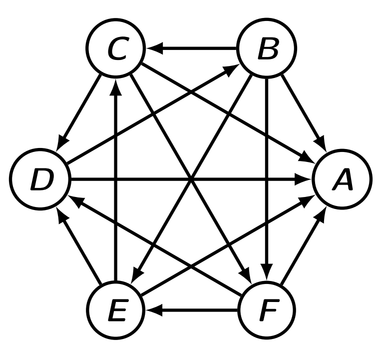
Problem 1.3 (2pt)**.** *Compute the number of different tournaments with n players. Two tournaments are different if they have different outcomes for at least one game.*

Definition 1.3. We call a tournament *transitive* if whenever there are distinct players p, q, r where p beats q and q beats r, then p beats r. A tournament with 1 or 2 players is always transitive.

Problem 1.4 (1pt)**.** *Construct a transitive tournament with 4 players.*

Problem 1.5 (4pt)**.** *How many different tournaments with n players are transitive? Justify your answer.*

Problem 1.6 (2pt)**.** *Seagulls love transitivity and will flip the results of some games in a tournament to make the overall tournament transitive. Flipping a game takes an edge $(p \to q)$ and turns it into $(q \to p)$. However, they are also lazy, so they will always flip the fewest number of games possible. The tournament depicted below just occurred and is untouched by the seagulls. Compute the number of games the seagulls will flip in order to make it transitive. List the games they flip. (There may be multiple correct answers.)*



Definition 1.4. A tournament is *acyclic* if for any player p, there is no sequence of edges starting from p that ends at p.

It turns out that a tournament being transitive and being acyclic are actually the same exact thing!

Problem 1.7 (3pt)**.** *Prove that if a tournament with $n \geq 2$ players is acyclic, then it is transitive.*

Problem 1.8 (3pt)**.** *Prove that if a tournament with $n \geq 2$ players is transitive, then it is acyclic.*

Problem 1.9 (7pt)**.** *Recall that the seagulls always flip the fewest number of edges necessary to make a tournament transitive. Another tournament with 6 players takes place. To the devastation of the seagulls, it turns out to be the worst-case scenario—there are no tournaments on 6 players that would require them to flip more games. How many edges do the seagulls need to flip in this worst case? Prove that no tournament with 6 players will ever require them to flip more than this number and that there exists a tournament where the seagulls must flip this many edges.*

Definition 1.5. Let $T = (V, E)$ be a tournament, and suppose $V' \subset V$ is nonempty. Let $E' \subset E$ be a set of directed edges such that if $p, q \in V'$ and $(p \to q) \in E$, then $(p \to q) \in E'$. We say that the resulting tournament $T' = (V', E')$ is a *subtournament* of T.

In other words, a subtournament contains a subset V' of all of the players, and the result of all the games between the players in V' are the same as they were in V. Subtournaments are often a good way to analyze tournaments when we only know a limited amount of information.

Definition 1.6. A tournament has a *cycle* if there exists any player p such that there is a nonzero sequence of distinct edges starting from p that ends at p. The length of the cycle is the number of distinct players it contains.

For example, Problem 1.6's tournament has a cycle starting and ending at B, namely $(B \to C), (C \to D), (D \to B)$.

Problem 1.10 (5pt)**.** *Prove that if a tournament has a cycle of length k, where $k \geq 3$, then the tournament has a cycle of length i for all $3 \leq i \leq k$.*

Unfortunately, transitive tournaments are hard to come by. Luckily, when we examine subtournaments, transitivity is a lot more common!

Problem 1.11 (2pt)**.** *Prove that any tournament with 2^2 players contains a transitive subtournament with 3 players.*

Problem 1.12 (6pt)**.** *Prove that for every positive integer n, any tournament with 2^n players contains a transitive subtournament with $n+1$ players. (A proof that earns **full** credit for this problem automatically earns you full credit for 1.11.)*

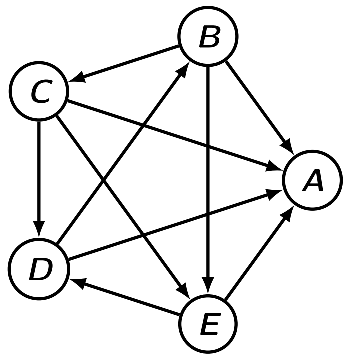
2 Scores Sequences and Reducibility

Understanding the underlying structure of tournaments all begins with the scores. Given a tournament, what can we say about the scores?

Definition 2.1. Let $T = (V, E)$ be an n-tournament (a tournament with n players) where $V = \{p_1, \ldots, p_n\}$. The *score* of each player p_i is denoted s_i and is given by the number of outgoing edges from p_i (the number of games p_i wins).

Definition 2.2. List the player scores of a tournament T in a nondecreasing order. The resulting list is called the *score sequence* of T, which we denote as S. A score sequence S is called valid if there exists a tournament where the scores of all the players, ordered in nondecreasing order, is exactly S.

Problem 2.1 (1pt). *Compute the score sequence for the following tournament.*



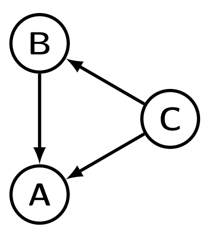
Problem 2.2 (2pt). *Find all possible score sequences for a transitive tournament with n people. Justify your answer.*

Definition 2.3. We call a tournament *reducible* if there is a way to partition V into two nonempty sets W and L such that the following two criteria are satisfied:

- Every player in V is in exactly one of W and L
- Every player in W beats every player in L.

If it is not possible to partition the players in such a way, then the tournament is *irreducible.*

For example, the tournament:



is reducible into two groups, $W = \{C\}$ and $L = \{A, B\}$, since C beats both A and B.

Problem 2.3 (1pt). *Construct a tournament of 5 people that is reducible into a group of* 3 *and a group of* 2. *List the two sets of players and label them with "winners" and "losers".*

Definition 2.4. A *random n-tournament* $\widetilde{T}_n$ is a tournament in which for every pair of players (p, q), the probability that p beats q is $\frac{1}{2}$.

Problem 2.4 (1pt). *Compute the probability that $\widetilde{T}_3$, a random* 3*-tournament, is reducible.*

Problem 2.5 (5pt). *Prove that if the absolute value of the difference in scores between all pairs of players of an n-tournament T_n is less than $\frac{n}{2}$, then T_n is irreducible.*

Cycles with length three are quite common in tournaments, and they're even more so in irreducible tournaments. One fun fact we won't prove here is that every player in an irreducible tournament is contained within some cycle!

Problem 2.6 (2pt). *Prove that every irreducible tournament with* 4 *players contains a cycle of length* 3.

Problem 2.7 (3pt)**.** *Prove that every irreducible tournament with $n \geq 4$ people contains a cycle of length 3. (A proof that earns **full** credit for this problem automatically earns you full credit for 2.6.)*

Definition 2.5. Let $T = (V, E)$ be a tournament. A *directed path* is a sequence of directed edges in E that joins a sequence of distinct vertices in V.

For example, the sequence $(E \to D), (D \to B), (B \to A)$ is a directed path in the tournament in Problem 2.1 that joins together the vertices E, D, B, and A.

Definition 2.6. We say an n-tournament $T = (V, E)$ has a *Hamiltonian path* if we can find some ordering of the players into the list $(p_1, \ldots, p_n)$ such that the distinct edges $(p_1 \to p_2), (p_2 \to p_3), \ldots, (p_{n-1} \to p_n)$ are in the set E.

In other words, a Hamiltonian path is a directed path that visits every player exactly once.

Problem 2.8 (2pt)**.** *Prove that every transitive tournament has a Hamiltonian path.*

The tournament doesn't even have to be transitive! In fact, we will prove that there is a Hamiltonian path in every tournament.

Problem 2.9 (4pt, 1pt)**.**

(a) Suppose you have a nonempty directed path that joins vertices $p_1, \ldots, p_i$. Given a player x that is not already in the path, prove that you can always create a path that joins together vertices $p_1, \ldots, p_i$ and x, in some order.

(b) Prove that every tournament has a Hamiltonian path.

Problem 2.10 (2pt)**.** *Are the following score sequences valid? If so, then you do not need to provide any justification. If not, then provide a brief explanation **without** explicitly referencing 2.11 or 2.12.*

1. *(1, 1, 2, 2)*
2. *(1, 1, 3, 3, 3, 4)*
3. *(3, 3, 3, 3, 3, 3, 3)*
4. *(1, 1, 1, 2, 4, 6, 6)*

One way to view the validity of score sequences is to remove the player that won the most games and check if the resulting score sequence is valid. Consider a non-decreasing sequence of n integers $(s_1, \ldots, s_n)$ where $m = s_n$. We can conclude that $(s_1, \ldots, s_n)$ is the score sequence of an n-tournament if and only if the new sequence $(s_1, s_2, ..., s_m, s_{m+1} - 1, ..., s_{n-1} - 1)$, when arranged in non-decreasing order, is a valid score sequence of some $(n-1)$-tournament. Using this idea, we examine a mathematical expression to check this same criterion.

For the following two questions, consider the following nondecreasing ordered set of integers $S = (s_1, \ldots, s_n)$.

Problem 2.11 (2pt)**.** *If S is a valid score sequence, prove that*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} = \frac{k(k-1)}{2}$$

for all $1 \leq k \leq n$, with equality at $k = n$.

Problem 2.12 (12pt)**.** *If S satisfies*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} = \frac{k(k-1)}{2}$$

for all $1 \leq k \leq n$, with equality at $k = n$, prove that S is a valid score sequence.

3 Who Wins?

Tournaments give rise to some interesting players. Let's take a deep dive into some of these special players.

Definition 3.1. We say a player k of a tournament T is a *king* of T if either k wins against all other players of the tournament, or for all players m such that k loses to m, there exists a third player p such that k wins against p and p wins against m.

Definition 3.2. We say a king a of a tournament T is *Alexandrian* if a beats all other players of T.

In other words, a king is a player who wins against everyone in either one or two steps, and an Alexandrian king is a special kind of king that beats everyone in exactly one step.

Problem 3.1 (1pt)**.** *Draw a* 3*-tournament with exactly* 1 *king and another with exactly* 3 *kings.*

Problem 3.2 (2pt)**.** *Show that no* 4*-tournament can have exactly* 2 *kings. (Do not apply Problem 3.7.)*

Definition 3.3. Let $T = (V, E)$ be a tournament, and say p is a player of T. Let $W^p \subseteq V$ be the set of all players that p wins against, and let $L^p \subseteq V$ be the set of all players that p loses against. Let E_W and E_L be the sets of edges that correspond to all games played by players in W^p and L^p respectively. (Refer to Definition 1.5 for a formal definition of these sets of edges.) We call the resulting subtournaments $W_p = (W^p, E_W)$ and $L_p = (L^p, E_L)$ the *winners with respect to* p and the *losers with respect to* p respectively.

Problem 3.3 (2pt, 3pt)**.** *(King Seagull Theorem)*

(a) Suppose a player k has the most wins in the tournament or is tied for the most wins, and for the sake of contradiction suppose that k is not a king—i.e. that there exists some player p such that k does not win against p in one or two steps. Argue that p must beat everyone in subtournament W_k.

(b) Show that $|W^p| > |W^k|$, and use this to prove that every tournament must have at least 1 *king.*

As we have shown, every tournament has at least 1 king, which is desirable should we try to define the "winner" of a tournament. We can also characterize another relationship between players and kings!

Problem 3.4 (1pt, 2pt)**.**

(a) Let $T = (V, E)$ be a tournament, and suppose $p \in V$ is a player that has lost to at least one other player. Explain why the nonempty subtournament L_p contains a king.

(b) Argue that any king of L_p is also a king of T.

As a corollary, it follows that everyone who has lost at least once must have also lost to a king. Note that this *does not* imply that everyone who wins is a king!

Problem 3.5 (1pt)**.** *Construct a* 4*-tournament such that there is a player p in the tournament has at least one win but is not a king.*

Unfortunately, Problem 3.4 implies that the "winner" of a tournament in our naive sense is generally not unique.

Problem 3.6 (3pt)**.** *Show that a king is unique if and only if they are an Alexandrian king.*

*(Note: For a proof of an "**if and only if**" statement, you need to show the implication in **both** directions for full credit. In this question, you need to show that any unique king is an Alexandrian king, and any Alexandrian king must be a unique king.)*

Of course, Alexandrian kings are quite rare (as we will soon prove), so again, we generally expect to see more than a single king. Even so, it seems like seagulls really hate having two kings.

Problem 3.7 (3pt)**.** *Prove that no tournament can have exactly* 2 *kings.*

Trying to count exactly how many kings a tournament has is quite difficult, but we can try to estimate how many kings a tournament may have.

Problem 3.8 (2pt)**.** *Show that the probability that a random n-tournament contains an Alexandrian king approaches* 0 *as $n \to \infty$. (You may use the fact that if $p(n)$ is a polynomial function in n and $q(n)$ is an exponential function in n with base greater than 1, then $\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$.)*

As our intuition suggests, Alexandrian kings are (very) rare, and they only get increasingly rare as the size of the tournament grows larger. What about kings?

Definition 3.4. Let p and q be players in a tournament T. We say that an ordered pair of players (p, q) is *non-dominating* if p loses against q and for all players $r \in L^q$, p loses against r.

Problem 3.9 (1pt)**.** *Let* $S = (\{p_1, p_2, p_3\}, \{(p_1 \to p_2), (p_1 \to p_3), (p_2 \to p_3)\})$ *be a* 3*-tournament. Identify all non-dominating pairs in* S*.*

Problem 3.10 (1pt)**.** *Do Problem 3.9 with a cyclic tournament* $S' = (\{q_1, q_2, q_3\}, \{(q_1 \to q_2), (q_2 \to q_3), (q_3 \to q_1)\})$.

As 3.9 and 3.10 show, Definition 3.4 is just an unwieldy way of encapsulating the fact that if such a non-dominating pair (p, q) exists, then p is not a king of T.

Problem 3.11 (3pt)**.** *Suppose that* (p, q) *is an ordered pair of players in a random* n*-tournament* $\widetilde{T}_n$*. Find the probability that* (p, q) *is non-dominating.*

Problem 3.12 (3pt)**.** *Use Problem 3.11 to generate a lower bound on the probability that every player of* $\widetilde{T}_n$ *is a king. Your bound should approach* 1 *as* $n \to \infty$*. (You do not need to prove that your bound approaches* 1*.)*

In the limit, we might expect all players of the tournament to have approximately the same number of wins, which tells us that we should have a very large number of kings. Combined with our naive perception of "winners", this has the unfortunate implication that for large tournament sizes, *almost every player is a "winner"*.

To top off this section, we take a look at another way to classify tournaments. This time, instead of trying to shrink tournaments, we look at an even bigger one!

Definition 3.5. We call a tournament $T = (V, E)$ *super-kingable* if there exists a tournament $X = (U, F)$ where T is a subtournament of X and the set of kings of X is exactly V.

Problem 3.13 (2pts)**.** *Prove that if a non-trivial tournament* T *is super-kingable, then the tournament* T *itself does not contain an Alexandrian king. (A tournament with a single player is considered trivial.)*

Problem 3.14 (6pts)**.** *Prove that if a non-trivial tournament* T *contains no Alexandrian king, then* T *is super-kingable.*