Preliminaries

Time Limit: 80 minutes of work time.

Maximum Score: 111

ATTENTION: You will submit this entire packet! (and only this packet)

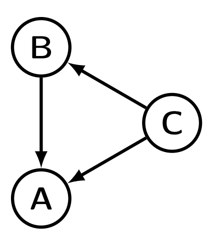
Instructions: Problems that use the words "compute", "list", "identify", or "draw" only call for an answer; no explanation or proof is needed. Unless otherwise stated, all other questions require explanation or proof.

- Each team will receive exactly one packet with answer boxes. This is the only packet that will be graded. The boxes are all appropriately sized for the length of the question's answer.
- Write answers in the given boxes for each question. Do **NOT** write on the back of any pages of the answer packet. Any work written on the back of packet pages will not be graded.
- Use blank scratch paper to work out problems first, then transfer your final answers into the boxes. You may grab more scratch paper as needed.
- If your team needs more room for a question, indicate this in the corresponding answer box **and** on the front page. Then, use one of the provided extra pages attached to the back of the packet to continue your answer. Label the answer with the question number. If you are writing multiple answers on an extra page, label each and separate them clearly using one horizontal line between questions.
- You may only use **one** side of each extra page (the side with text). If an extra page has work on two sides, neither side will be graded.
- Each team is limited to exactly **two** extra pages. Use them wisely!
- Each team will submit **all** 26 pages in the answer packet. They **must** be in the original page order. No other pages are allowed!

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them. The problems are ordered by content, **NOT DIFFICULTY**. It is to your advantage to attempt problems from throughout the test. While completing the round, you should not consult the internet or any materials outside of the content of this test (including results not covered in this power round). You may not use calculators.

Important: If a problem asks you to **construct** a tournament (or directed graph), you must neatly draw a directed graph with the proper vertices (enclosed in circles!) and **clearly** drawn directed edges (this means arrows!). If you use graphs in a proof, they **must** be in this format in order to be considered (using ellipses to denote an unknown number of edges is allowed). Make sure your graphs are large, clean, and legible.

As an example, here is the directed graph with vertices $\{A, B, C\}$ and edges $\{(B \to A), (C \to A), (C \to B)\}$.



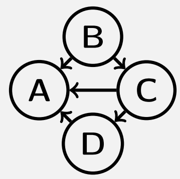
Team ID

Used extra page 1? Used extra page 2?

1 Tournaments and Transitivity

Definition 1.1. A *directed graph* is an ordered pair $G = (V, E)$ where V is a set containing elements called vertices and E is a set of ordered pairs of elements of V. Each element in E is called a directed edge. If $p, q \in V$ are chosen such that there exists a directed edge $(p, q) \in E$, we denote this directed edge as $(p \to q)$.

Problem 1.1 (1pt)**.** *Construct a directed graph with vertices* $\{A, B, C, D\}$ *and edges* $\{(D \to A), (B \to C), (C \to A), (C \to D), (B \to A)\}$.



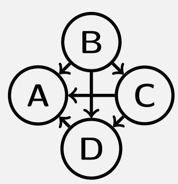
A natural way to model tournaments is to use directed graphs!

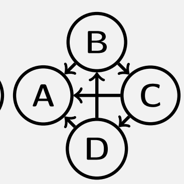
Definition 1.2. A *tournament* $T = (V, E)$ is a directed graph such that the vertices of V are called players, and there exists exactly one directed edge between every pair of players. In other words, if p and q are distinct players in T, then either $(p \to q)$ or $(q \to p)$. If $(p \to q)$, we say that p beats/wins against q.

The graph in Problem 1.1 is almost a tournament!

Problem 1.2 (1pt)**.** *Redraw the graph in problem 1.1 and modify it to make it a tournament. (There are multiple possible constructions.)*

Add either $(B \to D)$ or $(D \to B)$.





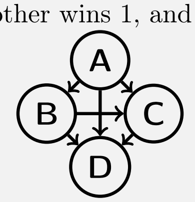
Problem 1.3 (2pt)**.** *Compute the number of different tournaments with n players. Two tournaments are different if there is at least one game with a different outcome.*

In a tournament with n players, there are a total of $\binom{n}{2}$ games. For each game, there are 2 options for winners which means there are a total of $2^{\binom{n}{2}}$ tournments.

Definition 1.3. We call a tournament *transitive* if whenever there are distinct players p, q, r where p beats q and q beats r, then p beats r. A tournament with 1 or 2 players is always transitive.

Problem 1.4 (1pt)**.** *Construct a transitive tournament with* 4 *players.*

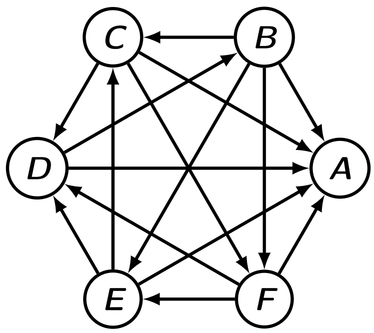
One player wins 3 games, another wins 2, another wins 1, and the last wins 0.



Problem 1.5 (4pt)**.** *How many different tournaments with n players are transitive? Justify your answer.*

Sol 1. Notice that for any pair of players, the number of wins they have is different. If player x beats player y, then x must have more wins than y by transitivity. Therefore, we can order the players by the number of wins they have.
Sol 2. Given the criteria for transitivity, notice that it is possible to find a player that beats all other players. First, since it is transitive, given any 3 players, one of the players beats the other two. Then, start out with any 3 players, keeping the "best" player, and swapping out the other two with new players. Repeat the swapping of new players until all the players have been in the group of 3 at least once. Then, the best player has been found. Remove this player and repeat to find all next best. This can be done until all the players have been ranked.
From above, we know that the ordering of the players directly tells us the result of every game (the higher ranked player in any pair won the game). Therefore, since there are $n!$ ways to rank the players, there are also $n!$ distinct transitive tournaments.

Problem 1.6 (2pt)**.** *Seagulls love transitivity and will flip the results of some games in a tournament to make the overall tournament transitive. Flipping a game takes an edge $(p \to q)$ and turns it into $(q \to p)$. However, they are also lazy, so they will always flip the fewest number of games possible. The tournament depicted below just occurred and is untouched by the seagulls. Compute the number of games the seagulls will flip in order to make it transitive. List the games they flip. (There may be multiple correct answers.)*



Flip $(D \to B)$ and $(E \to C)$ or $(C \to F)$ or $(F \to E)$.

Definition 1.4. A tournament is *acyclic* if for any player p, there is no sequence of edges starting from p that ends at p.

It turns out that being transitive and being acyclic are actually the same exact thing!

Problem 1.7 (3pt)**.** *Prove that if a tournament with $n \geq 2$ players is acyclic, then it is transitive.*

A tournament with 2 players is always acyclic and vacuously is transitive. When the tournament has at least 3 players, for any 3 players x, y and z, if x beats y and y beats z, we know that either x beats z or z beats x. Since the tournament is acyclic, we cannot have that z beats x, as there would be the cycle $x \to y \to z \to x$. Therefore, the result of the game must be that x beats z, which implies that the tournament is transitive by definition.

Problem 1.8 (3pt)**.** *Prove that if a tournament with $n \geq 2$ players is transitive, then it is acyclic.*

Assume for the sake of contradiction that T contains a directed cycle. Let this cycle be

$$p_1 \to p_2 \to \cdots \to p_k \to p_1$$

for some $k \geq 2$. By the transitivity of T, since we have $p_1 \to p_2$ and $p_2 \to p_3$, we must have $p_1 \to p_3$. Repeating this logic, we conclude that p_1 wins against all players in the cycle, including p_k. However, since $p_1 \to p_k$ and $p_k \to p_1$ cannot simultaneously exist, we have reached a contradiction, which implies that the graph must be acyclic.

Problem 1.9 (7pt)**.** *Recall that the seagulls always flip the fewest number of edges necessary to make a tournament transitive. Another tournament with 6 players takes place. To the devastation of the seagulls, it turns out to be the worst-case scenario—requiring them to flip more edges than in any other possible tournament on 6 players. How many edges do the seagulls need to flip in this worst case? Prove that no tournament on 6 players will ever require them to flip more than this number, and there exists a tournament where the seagulls must flip this many edges.*

Since a transitive tournament is acyclic, this problem is equivalent to finding the minimum number of edges to reverse in order to make the graph acyclic. We claim that 4 flips are needed.
We first show that 4 flips always suffice. We start with a claim that lets us handle one player at a time.
Claim 1. Let T be an n-tournament and let v be a player of T with score s_v. If the subtournament $T - v$ obtained by deleting v can be made transitive in m flips, then T can be made transitive in at most $\min(n - 1 - s_v,\ s_v) + m$ flips.

Proof. Flip the $n - 1 - s_v$ edges pointing into v, so that v becomes an Alexandrian king; alternatively, flip the s_v edges pointing out of v, so that v loses every game. Neither set of flips involves an edge of $T - v$, so we may spend m further flips making $T - v$ transitive. Now take any three players x, y, z with $x \to y$ and $y \to z$. If none of them is v, then $x \to z$ since $T - v$ is transitive. If v is an Alexandrian king, the only such triples containing v have $x = v$, and indeed $v \to z$; if v loses every game, the only such triples containing v have $z = v$, and indeed $x \to v$. Hence the whole tournament is transitive. □

Claim 2. Any 3-tournament and any 4-tournament can be made transitive in at most 1 flip.

Proof. A 3-tournament is either transitive or a 3-cycle, and flipping any edge of a 3-cycle makes it transitive. For a 4-tournament, the scores sum to $\binom{4}{2} = 6$. If some player has score 3, it is an Alexandrian king, and Claim 1 applied to that player gives $0 + 1 = 1$ flip; if some player has score 0, Claim 1 applied to that player gives $0 + 1 = 1$ flip. Otherwise every score is 1 or 2, so the score sequence is $(1, 1, 2, 2)$. Let a and b be the players of score 2 and assume without loss of generality that $a \to b$. Then b must beat both players of score 1, so a beats exactly one of them, say $a \to c$ and $d \to a$. Since d already has one win, $c \to d$. Flipping $(d \to a)$ yields the transitive tournament in which a beats everyone, b beats c and d, and c beats d. □

Claim 3. Any 5-tournament can be made transitive in at most 3 flips, and in at most 2 flips if its score sequence is not $(2, 2, 2, 2, 2)$.

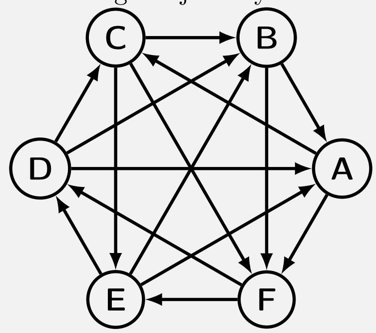
Proof. The scores sum to $\binom{5}{2} = 10$, so the largest score m is at least 2, and $m = 2$ forces the score sequence $(2, 2, 2, 2, 2)$. Applying Claim 1 to a player of score m, making it an Alexandrian king, together with Claim 2, takes at most $(4 - m) + 1$ flips. This is at most 2 when $m \geq 3$ and at most 3 when $m = 2$. □

Now let T be any 6-tournament. Its scores sum to $\binom{6}{2} = 15$, so the largest score is at least 3.

- (Case 1) Some player has score 4 or 5. Applying Claim 1 to that player, making it an Alexandrian king, together with Claim 3, takes at most $(5 - 4) + 3 = 4$ flips.
- (Case 2) The largest score is 3. Since the scores sum to 15 and none exceeds 3, the score sequence is $(0, 3, 3, 3, 3, 3)$, $(1, 2, 3, 3, 3, 3)$, or $(2, 2, 2, 3, 3, 3)$. In the first two cases, apply Claim 1 to a player v of smallest score, flipping the $s_v \leq 1$ edges out of v so that v loses every game; together with Claim 3 this takes at most $1 + 3 = 4$ flips.
  In the last case, let x, y, z be the three players of score 2. The three games among them produce three wins in total, so one of them, say v, wins at most one of these games; in particular v loses to another player u of score 2. Deleting v lowers the score of each player that beat v by exactly 1, so u has score 1 in $T - v$ and the score sequence of $T - v$ is not $(2, 2, 2, 2, 2)$. Applying Claim 1 to v, flipping the two edges out of v, together with Claim 3, takes at most $2 + 2 = 4$ flips.

In every case 4 flips suffice, so the seagulls never need to flip more than 4 edges.

The graph below gives a construction with 4 edge disjoint cycles.



Definition 1.5. Let $T = (V, E)$ be a tournament, and suppose $V' \subset V$ is nonempty. Let $E' \subset E$ be a set of directed edges such that if $p, q \in V'$ and $(p \to q) \in E$, then $(p \to q) \in E'$. We say that the resulting tournament $T' = (V', E')$ is a *subtournament* of T.

In other words, a subtournament contains a subset V' of all of the players, and the result of all the games between the players in V' are the same as they were in V. Subtournaments are often a good way to analyze tournaments when we only know a limited amount of information.

Definition 1.6. A tournament has a *cycle* if there exists any player p such that there is a nonzero sequence of distinct edges starting from p that ends at p. The length of the cycle is the number of distinct players it contains.

For example, Problem 1.6's tournament has a cycle starting and ending at B, namely $(B \to C), (C \to D), (D \to B)$.

Problem 1.10 (5pt)**.** *Prove that if a tournament has a cycle of length k, where $k \geq 3$, then the tournament has a cycle of length i for all $3 \leq i \leq k$.*

Assume the cycle consists of the players $1 \to \ldots \to k$. Consider the subtournment consisting of players 1 through k. Since this subtournament is not transitive, as there is no player that wins zero times, we can find three players x, y, z where x beats y, y beats z, and z beats x, which forms a cycle of length 3. Suppose we have a cycle of players $u_1, \ldots, u_i$. If there is a player x outside of the cycle where some player u_k beats x and x beats u_{k+1}, then we can insert x into the cycle between u_k and u_{k+1}. We can also insert x if u_i beats x and x beats u_1.
Otherwise, then the vertices outside the cycle come in two types. Let us denote the set of players in the cycle as C. There is a set A of players such that all players in C beat all players in A. Also, there is a set B of vertices such that all the players in B beat all the players in C.
Because the subtournament contains a cycle of length k, which means we can reach any player z from any player y, there must be a path leaving cycle C to go to a player outside C, but then coming back to C. That path must go from C to A (it can't go to B) but it can only come back to C from B, so eventually it must go from A to B somehow. In particular, there must be an edge $(a \to b)$ with $a \in A$ and $b \in B$. Now, we can extend C pretty much any way we like. One way to get a cycle of length $i+1$ is $(u_1 \to a \to b \to u_3 \to \ldots \to u_k \to u_1)$. Since we know we have a cycle of 3 players, we can continue extending the cycle using this logic one by one and obtain cycles of all lengths from 3 to k.

Unfortunately, transitive tournaments are hard to come by. Luckily, when we examine subtournaments, transitivity is a lot more common!

Problem 1.11 (2pt)**.** *Prove that any tournament with 2^2 players contains a transitive subtournament with 3 players.*

Consider any tournament with 4 players. We note that there are 4 total subtournaments with 3 players, formed by removing one of the four players. Assume for the sake of contradiction that every subtournament of 3 players is cyclic. Without loss of generality, assume we have two players a and b where $a \to b$. Then, in order for the subtournaments with (a, b, c) and (a, b, d) to be cyclic, we need both $c \to a$ and $d \to a$. However, this means it is impossible for the subtournament of (a, c, d) to be cyclic, a contradiction. Thus, any tournament with 4 players must contain a transitive subtournament with 3 players.

Problem 1.12 (6pt). *Prove that for every positive integer n, any tournament with 2^n players contains a transitive subtournament with $n+1$ vertices. (A proof that earns **full** credit for this problem automatically earns you full credit for 1.11)*

We claim that for all n, a tournament with 2^n players contains a transitive subtournament with $n+1$ vertices. The base case of $n = 1$ is clearly true since a tournament with two players is always transitive. Assume for the sake of induction that the claim holds for some k. Then, consider any tournament with 2^{k+1} players. Pick any player u. Then, split the remaining $2^{k+1} - 1$ players into two groups, namely the players that win against u in group W and the players that lose against u in a group L. By the Pigeonhole Principle, one of the groups must contain at least 2^k players. We conclude that there exists a transitive subtournament T_k with $k + 1$ players in this group. If this group is W, then we can add in u to the T_k as the worst player, resulting in a transitive tournament with $k + 2$ players. If this group is L, then we can add in u to T_k as the best player, which also results in a transitive tournament with $k + 2$ players. In either case, we can find a subtournament of $k + 2$ players that is transitive. By induction, we are done.

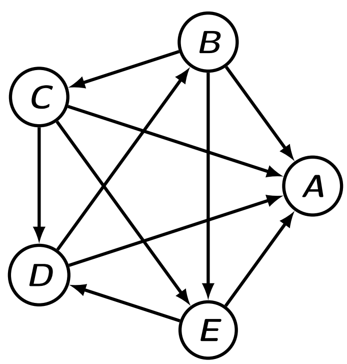
2 Scores Sequences and Reducibility

Understanding the underlying structure of tournaments all begins with the scores. Given a tournament, what can we say about the scores?

Definition 2.1. Let $T = (V, E)$ be an n-tournament (a tournament with n players) where $V = \{p_1, \ldots, p_n\}$. The *player score* of each player p_i is denoted s_i and is given by the number of outgoing edges from p_i (the number of games p_i wins).

Definition 2.2. List the player scores in a nondecreasing order. The resulting list is called the *score sequence*, which we denote as S. A score sequence S is called *valid* if there exists a tournament where the scores of all the players, ordered in nondecreasing order, is exactly S.

Problem 2.1 (1pt). *Compute the score sequence for the following tournament.*



The correct answer is $(0, 2, 2, 3, 3)$.

Problem 2.2 (2pt). *Find all possible score sequences for a transitive tournament with n people. Justify your answer.*

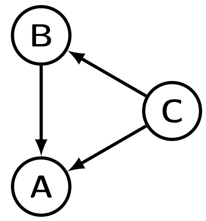
The only score sequence is $(0, 1, \ldots, n-1)$. This is a direct result of 1.6 as each person wins a different number of games.

Definition 2.3. We call a tournament *reducible* if there is a way to partition V into two nonempty sets W and L such that the following two criteria are satisfied:

- Every player in V is in exactly one of W and L
- Every player in W beats every player in L.

If it is not possible to partition the players in such a way, then the tournament is *irreducible*.

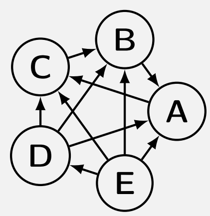
For example, the tournament:



is reducible into two groups, $W = \{C\}$ and $L = \{A, B\}$, since C beats both A and B.

Problem 2.3 (1pt)**.** *Construct a tournament of 5 people that is reducible into a group of* 3 *and a group of* 2. *List the two sets of players and label them with "winners" and "losers".*

One possible correct answer could be:



where the winners are (D, E) and the losers are (A, B, C).

Definition 2.4. A *random n-tournament* $\widetilde{T}_n$ is a tournament in which for every pair of players (p, q), the probability that p beats q is $\frac{1}{2}$.

Problem 2.4 (1pt)**.** *Compute the probability that* $\widetilde{T}_3$, *a random* 3*-tournament, is reducible.*

There are 8 possible tournaments. Two of them are cyclic and are irreducible. The remaining 6 are reducible, so the probability is $\frac{3}{4}$.

Problem 2.5 (5pt)**.** *Prove that if the absolute value of the difference between any two players of an n-tournament* T_n *is less than* $\frac{n}{2}$, *then* T_n *is irreducible.*

Notice that we are only concerned about the players with the most and the least wins. Assume for the sake of contradiction that T_n is reducible. Then, there exists a way to partition the set of n players into nonempty sets of winners W and losers L such that all players in W win against all players in L. Let L contain k players and W contain $n - k$ players where $n > k \geq 1$. Notice that the player with the least wins must be in L and the player with the most wins must be in W. Then, the maximum number of wins that the player with the least wins in L could have is $\left\lfloor \frac{k-1}{2} \right\rfloor$ by the Pigeonhole Principle, since the players in L win only against other players in L and there are a total of $\binom{k}{2}$ such wins split amongst k players. Additionally, the minimum number of games that the player with the most wins could have just against other players in W is $\left\lceil \frac{n-k-1}{2} \right\rceil$ games won against other players in W by the Pigeonhole Principle, since there are a total of $\binom{n-k}{2}$ wins split amongst $n - k$ players. This implies that there exists a player with at least $\left\lceil \frac{n-k-1}{2} \right\rceil + k$ total wins, with the k wins from the k players in L. Therefore, a lower bound for the difference is

$$\left\lceil \frac{n-k-1}{2} \right\rceil + k - \left\lfloor \frac{k-1}{2} \right\rfloor \geq \frac{n-k-1}{2} + k - \frac{k-1}{2} = \frac{n}{2},$$

contradicting the assumption that the difference between any two players is less than $\frac{n}{2}$, which implies that T_n must be irreducible.

Cycles with length three are quite common in tournaments, and they're even more so in irreducible tournaments. One fun fact we won't prove here is that every player in an irreducible tournament is contained within some cycle!

Problem 2.6 (2pt)**.** *Prove that every irreducible tournament with* 4 *players contains a cycle of length* 3.

Assume for the sake of contradiction that there are no 3-cycles. This implies that all 4 subtournaments with 3 people are transitive. Without loss of generality, let one of the transitive subtournaments consist of players $1, 2$, and 3 (ranked in that order as well). Player 4 cannot lose to 1 since then the tournament would be reducible. Moreover, 4 cannot win against both 2 and 3 since then the tournament would be reducible. Thus one of 2 or 3 must beat 4, which creates a cycle of length 3.

Problem 2.7 (3pt)**.** *Prove that every irreducible tournament with $n \geq 4$ people contains a cycle of length* 3. *(A proof that earns **full** credit for this problem automatically earns you full credit for 2.6.)*

We prove the contrapositive, namely, if a tournament has no 3-cycles, then it is reducible. Recall that a tournament is transitive if and only if for every (x, y, z) if $x \to y$ and $y \to z$, then $x \to z$. Notice that this is equivalent to the tournament having no 3-cycles, as any 3-cycle immediately breaks transitivity. Hence, any tournament without 3-cycles must be transitive and therefore reducible since there exists a player w who beat everyone else, so we can paritition the players into two groups, one with w and one with all the others.

Definition 2.5. Let $T = (V, E)$ be a tournament. A *directed path* is a sequence of directed edges in E that joins a sequence of distinct vertices in V.

For example, the sequence $(E \to D), (D \to B), (B \to A)$ is a directed path in the tournament in Problem 2.1 that joins together the vertices E, D, B, and A.

Definition 2.6. We say an n-tournament $T = (V, E)$ has a *Hamiltonian path* if we can find some ordering of the players into the list $(p_1, \ldots, p_n)$ such that the distinct edges $(p_1 \to p_2), (p_2 \to p_3), \ldots, (p_{n-1} \to p_n)$ are in the set E.

In other words, a Hamiltonian path is a directed path that visits every player exactly once.

Problem 2.8 (2pt)**.** *Prove that every transitive tournament has a Hamiltonian path.*

Since the players in a transitive tournament can be ranked with no ties, without loss of generality, assume player i is the ith best player. Then, we have a Hamiltonian path, namely $1 \to 2 \ldots \to n$.

The tournament doesn't even have to be transitive! In fact, we will prove that there is a Hamiltonian path in every tournament.

Problem 2.9 (4pt, 1pt)**.**

(a) Suppose you have a nonempty directed path that joins vertices $p_1, \ldots, p_i$. Given a player x that is not already in the path, prove that you can always create a path that joins together vertices $p_1, \ldots, p_i$ and x, in some order.

(b) Prove that every tournament has a Hamiltonian path.

Assume for the sake of contradiction that we cannot extend the path with x. Then, in order to prevent us from being able to add in x to the current path, we must have that u_1 wins against x. It follows that p_{i-1} must win against x. Then, p_i must win and lost against x in order for it to not be added to the path, which is impossible. Therefore x can always be added into the path, extending it.
Start out with any player. Then, use the procedure outlined in 2.11 to continue adding new players to the path until all players have been used. The resulting path is a Hamiltonian path.

Problem 2.10 (2pt)**.** *Are the following score sequences valid? If so, you do not need to provide any justification. If not, provide a brief explanation **without** referencing 2.11 or 2.12.*

1. *(1,1,2,2)*
2. *(1,1,3,3,3,4)*
3. *(3,3,3,3,3,3,3)*
4. *(1,1,1,2,4,6,6)*

Yes, yes, yes, no (there are two players that win against all other players, which is impossible).

One way to view the validity of score sequences is to remove the player that won the most games and check if the resulting score sequence is valid. Consider a non-decreasing sequence of n integers $(s_1, \ldots, s_n)$ where $m = s_n$. We can conclude that $(s_1, \ldots, s_n)$ is the score sequence of an n-tournament if and only if the new sequence $(s_1, s_2, ..., s_m, s_{m+1} - 1, ..., s_{n-1} - 1)$, when arranged in non-decreasing order, is a valid score sequence of some $(n-1)$-tournament. Using this idea, we examine a mathematical expression to check this same criterion.

For the following two questions, consider the following nondecreasing ordered set of integers $S = (s_1, \ldots, s_n)$.

Problem 2.11 (2pt)**.** *If S is a valid score sequence, prove that*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} = \frac{k(k-1)}{2}$$

for any $1 \leq k \leq n$, with equality at $k = n$.

If S is a valid score sequence, then the sum of the first k, when $k < n$, scores is equal to the total number of wins by the worst k players. Each one of these k players has played against all the other worst $k-1$ players, which means the must have won at least $\binom{k}{2}$ games. Any of the k players could have won against any of the $n-k$ remaining players as well, so the total sum can be greater than $\binom{k}{2}$. The sum of all n scores must be $\binom{n}{2}$ since there are a total of $\binom{n}{2}$ games.

Problem 2.12 (12pt)**.** *If S satisfies*

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2} = \frac{k(k-1)}{2}$$

for any $1 \leq k \leq n$, with equality at $k = n$, prove that S is a valid score sequence.

We claim that for any n, if a score sequence $(s_1, \dots, s_n)$ satisfies the set of inequalities, then it is a valid score sequence. The base case of two players is true, the only score sequence that satisfies the condition is $(0, 1)$, which is the only valid score sequence for two players. Now, assume that the claim is true for some k and consider some score sequence for $k+1$ players, $(s_1, \dots, s_{k+1})$ that satisfies the inequalities. Let's call a sequence satisfying this set of properties as *good*. We give a recursive algorithm proving that every good sequence is also a valid score sequence.

ConstructTournament(S):

- If $n \triangleq |S| = 1$, just return the solo player tournament.
- If there exists $1 \le k \le n-1$ such that $\sum_{i=1}^{k} s_i = \binom{k}{2}$:
  - Let $T_1 = \textbf{ConstructTournament}(s_1, s_2, \dots, s_k)$
  - Let $T_2 = \textbf{ConstructTournament}(s_{k+1} - k, \dots, s_n - k)$
  - Let T be the tournament consisting of all edges in T_1, all edges in T_2, and edges $i \to j$ for all $k+1 \le i \le n$, $1 \le j \le k$, and return T.
- Else:
  - Let $m = s_n$.
  - Let $T = \textbf{ConstructTournament}(s_1, s_2, \dots, s_m, s_{m+1} - 1, s_{m+2}, \dots, s_{n-1}, s_n + 1)$.
  - Flip the $(m+1, n)$ edge and return T.

We prove that this algorithm terminates and always constructs valid score sequence, by structural induction on the structure of the sequence S (in addition, we prove a slightly stronger property: player n beats the weakest s_n players, where weakest is by number of wins). In the base case, $S = (0)$ terminates and constructs the right sequence. Otherwise, we case on whether we end up in the if or else branch.

- Suppose we are in the if case. Then, by structural induction (as $k < n$), since the first k values form a good sequence, T_1 is immediately a valid tournament.
  However, it is less clear that T_2 is a good sequence. To prove that it is, we claim that $\sum_{i=k+1}^{k} s_i - k \ge \binom{j-k}{2}$ for all $j \ge k+1$. Indeed, note that

$$\binom{j}{2} \le \sum_{i=1}^{k} s_i = \binom{k}{2} + \sum_{i=k+1}^{k} s_i = \binom{k}{2} + k(j-k) + \sum_{i=k+1}^{j} s_i - k \implies$$

$$\begin{aligned}\sum_{i=k+1}^{j} s_i - k &\ge \binom{j}{2} - \binom{k}{2} - k(j-k) \\ &= \frac{1}{2}\left(j^2 - j - k^2 + k - 2kj + 2k^2\right) \\ &= \frac{1}{2}\left((j-k)^2 - (j-k)\right) \\ &= \binom{j-k}{2}\end{aligned}$$

  which is as desired. Furthermore, at $j = n$ all of the prior inequalities are in fact equalities.
  Therefore, $(s_{k+1} - k, s_{k+2} - k, \dots, s_n - k)$ is a good sequence. This implies by structural induction (as $n - k < n$) that T_2 is indeed a valid tournament. Then, each player in T_2 gains back exactly k wins, which thus implies that T is a valid tournament with score sequence S. Furthermore, we have that player n beats all of the first k players and the weakest $s_n - k$ players in T_2, so therefore beats the weakest s_n players.
  To show termination, both T_1 and T_2 have a smaller $(|S|, |S| - 1 - s_n)$ value which implies that the algorithm must terminate.
- Suppose now that we are in the else case. Note that the recursive call must increase s_n, which is bounded above by $n-1$, so such a call decreases the variant $(n, n-1-s_n)$ and therefore we have termination (furthermore, if $s_n = n-1$ then $\sum_{i=1}^{n-1} s_i = \binom{n-1}{2}$ so the Else case is unreachable).
  We wish to show that our recursive call results in a good sequence. Since $\sum_{i=1}^{k} s_i > \binom{k}{2}$ for all i, decreasing s_{m+1} by 1 cannot make $\sum_{i=1}^{k} s_i < \binom{k}{2}$ for any i (each of these partial sums can decrease by at most 1). Furthermore, it follows that player n must beat the first $m+1$ players in T and thus in particular beats player $m+1$. Therefore, flipping the result of the $(m+1, n)$ match implies that the resulting tournament is exactly the one for S.

Hence, we are done by structural induction.

3 Who Wins?

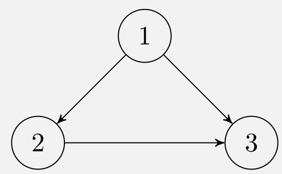
Tournaments give rise to some interesting players. Let's take a deep dive into some of these special players.

Definition 3.1. We say a player k of a tournament T is a *king* of T if either k wins against all other players of the tournament, or for all players m such that k loses to m, there exists a third player p such that k wins against p and p wins against m.

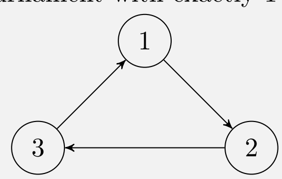
Definition 3.2. We say a king a of a tournament T is *Alexandrian* if a beats all the players of T.

In other words, a king is a player who wins against everyone in either one or two steps, and an Alexandrian king is a special kind of king that beats everyone in exactly one step.

Problem 3.1 (1pt)**.** *Draw a* 3*-tournament with exactly* 1 *king and another with exactly* 3 *kings.*



3-tournament with exactly 1 king.



3-tournament with exactly 3 kings.

Problem 3.2 (2pt)**.** *Show that no* 4*-tournament can have exactly* 2 *kings. (Do not apply Problem 3.7.)*

Suppose there exists $T = (\{k_1, k_2, p_3, p_4\}, \{e_1, \ldots, e_6\})$ such that k_1 and k_2 are kings of T. For concreteness assume $k_1 \to k_2$. Now k_2 must beat k_1 in exactly two steps, so say $k_2 \to p_1$ and $p_1 \to k_1$. Note that $p_1 \not\to p_2$, because then p_1 is a king of T. Hence $p_2 \to p_1$. But then $p_2 \not\to k_2$, becuase then p_2 is a king of T. Now if $k_1 \to p_2$ then p_1 is a king of T, but if $p_2 \to k_1$ then p_2 is a king of T. Thus the construction is impossible.

Definition 3.3. Let $T = (V, E)$ be a tournament, and say p is a player of T. Let $W^p \subseteq V$ be the set of all players that p wins against, and let $L^p \subseteq V$ be the set of all players that p loses against. Let E_W and E_L be the sets of edges that correspond to all games played by players in W^p and L^p respectively. (Refer to Definition 1.5 for a formal definition of these sets of edges.) We call the resulting subtournaments $W_p = (W^p, E_W)$ and $L_p = (L^p, E_L)$ the *winners with respect to* p and the *losers with respect to* p respectively.

Problem 3.3 (2pt, 3pt)**.** *(King Seagull Theorem)*

(a) Suppose a player k has the most wins in the tournament or is tied for the most wins, and for the sake of contradiction suppose that k is not a king—i.e. that there exists some player p such that k does not win against p in one or two steps. Argue that p must beat everyone in W_k.

(b) Show that $|W^p| > |W^k|$, and use this to prove that every tournament must have at least 1 *king.*

(a) Suppose otherwise. Then there exists some $q \in W_k$ such that $k \to q$ and $q \to p$, a contradiction.
(b) From part (a), $W^k \subseteq W^p$, and noting that $p \to k$, we see that $W^k \subset W^p$ is proper. Hence $|W^p| > |W^k|$, which contradicts that k has the most wins in the tournament, so k must be a king.

As we have shown, every tournament has at least 1 king, which is desirable should we try to define the "winner" of a tournament. However, we can say more: generally, a tournament has many kings, by the following interesting fact.

Problem 3.4 (1pt, 2pt)**.**

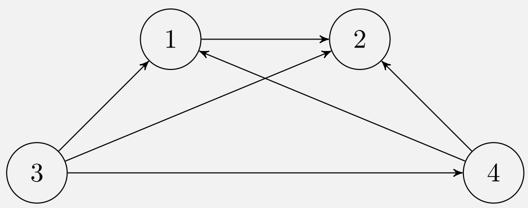
(a) Let $T = (V, E)$ be a tournament, and suppose $p \in V$ is a player that has lost to at least one other player. Explain why the nonempty subtournament L_p contains a king.

(b) Argue that any king of L_p is also a king of T.

(a) This is a direct application of the King Seagull Theorem.
(b) Let k be a (possibly unique) king of L_p. Note that $\{p, W^p, L^p\}$ form a partition of $V(T)$. By definition, $k \to p$, and since p beats everyone in W_p, k must beat everyone in W_p in at most two steps. Thus k is a king of T.

As a corollary, it follows that everyone who has lost at least once must have also lost to a king. Note that this *does not* imply that everyone who wins is a king!

Problem 3.5 (1pt)**.** *Construct a 4-tournament such that a player p in the tournament has at least one win but is not a king.*



Vertex 1 has at least one win but is not a king.

Unfortunately, Problem 3.4 implies that the "winner" of a tournament in our naive sense is generally not unique.

Problem 3.6 (3pt)**.** *Show that a king is unique if and only if they are an Alexandrian king.*

*(Note: For a proof of an "**if and only if**" statement, you need to show the implication in **both** directions for full credit. In this question, you need to show that any unique king is an Alexandrian king, and any Alexandrian king must be a unique king.)*

($\Rightarrow$) Suppose that k is a unique king, and say for the sake of contradiction that k is not an Alexandrian king (i.e. k does not beat everyone). Then k loses to at least one player; in particular, k must lose to a king by Problem 3.4, a contradiction.
($\Leftarrow$) Let e be an Alexandrian king. Clearly e is a king, and moreover, no player beats e. Then no one can beat e in either one nor two steps, so e is a unique king.

Of course, Alexandrian kings are quite rare (as we will soon prove), so again, we generally expect to see more than a single king.

Problem 3.7 (3pt)**.** *Prove that no tournament can have exactly 2 kings.*

Say k_1 and k_2 are the only kings of a tournament T, and consider the subtournament L_{k_2}. It must be nonempty, because k_1 must have beaten k_2 in one or two steps. But then k_2 must have lost to a king by Problem 3.4; hence, $k_1 \to k_2$. The same argument in reverse shows that $k_2 \to k_1$, which is impossible.

Trying to count exactly how many kings a tournament has is quite difficult, but we can try to estimate how many kings a tournament may have.

Problem 3.8 (2pt)**.** *Show that the probability that a random n-tournament contains an Alexandrian king approaches 0 as $n \to \infty$. (You may use the fact that if $p(n)$ is a polynomial function in n and $q(n)$ is an exponential function in n with base greater than 1, then $\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$.)*

Index each player of the tournament as $p_1, \dots, p_n$. The probability that p_j is an Alexandrian king can be computed by considering that each directed edge containing p_j must satisfy $(p_j \to \cdot)$. Since there are $n-1$ other members,

$$\mathbb{P}(\{p_j \text{ is an Alexandrian king}\}) = \frac{1}{2^{n-1}}.$$

By symmetry, the desired probability is

$$\mathbb{P}(\{\widetilde{T}_n \text{ contains an Alexandrian king}\}) = \sum_{j=1}^{n} \mathbb{P}(\{p_j \text{ is an Alexandrian king}\}) = \frac{n}{2^{n-1}} \xrightarrow{n\to\infty} 0.$$

As our intuition suggests, Alexandrian kings are (very) rare, and they only get increasingly rare as the size of the tournament grows larger. What about kings?

Definition 3.4. Let p and q be players in a tournament T. We say that an ordered pair of players (p, q) is *non-dominating* if p loses against q and for all players $r \in L^q$, p loses against r.

Problem 3.9 (1pt)**.** *Let $S = (\{p_1, p_2, p_3\}, \{(p_1 \to p_2), (p_1 \to p_3), (p_2 \to p_3)\})$ be a 3-tournament. Identify all non-dominating pairs in S.*

The ordered pairs (p_2, p_1), (p_3, p_1), and (p_3, p_2) are non-dominating.

Problem 3.10 (1pt)**.** *Do Problem 3.9 with a cyclic tournament $S' = (\{q_1, q_2, q_3\}, \{(q_1 \to q_2), (q_2 \to q_3), (q_3 \to q_1)\})$.*

There are no non-dominating pairs in S'.

As 3.9 and 3.10 show, Definition 3.4 is just an unwieldy way of encapsulating the fact that if such a non-dominating pair (p, q) exists, then p is not a king of T.

Problem 3.11 (3pt)**.** *Suppose that (p, q) is an ordered pair of players in a random n-tournament $\widetilde{T}_n$. Find the probability that (p, q) is non-dominating.*

Let us consider an arbitrary member of the tournament r such that $r \neq p$ and $r \neq q$. Observe that (p, q) is non-dominating if and only if (1) $q \to p$ and (2) $q \to r$ and $p \to r$ or $r \to q$. The probability of event (1) is $\frac{1}{2}$, and the probability of event (2) is $\frac{3}{4}$. Since the outcomes of each match in a tournament are independent, we see that events (1) and (2) are independent for any choice of r. Then

$$\mathbb{P}(\{(p, q) \text{ is non-dominating}\}) = \frac{1}{2}\left(\frac{3}{4}\right)^{n-2}.$$

Problem 3.12 (3pt)**.** *Use Problem 3.11 to generate a lower bound on the probability that every player of $\widetilde{T}_n$ is a king. Your bound should approach 1 as $n \to \infty$. (You do not need to prove that your bound approaches 1.)*

There are $n(n-1)$ distinct ordered pairs $(p, q) \in V(\widetilde{T}_n) \times V(\widetilde{T}_n)$. Hence, the probability that not every member is a king of $\widetilde{T}_n$ is bounded above by the expression

$$n(n-1)\frac{1}{2}\left(\frac{3}{4}\right)^{n-2} = n(n-1)\frac{8}{9}\left(\frac{3}{4}\right)^{n}.$$

The desired lower bound is therefore

$$\mathbb{P}(\{\text{all members of } \widetilde{T}_n \text{ are kings}\}) \geq 1 - n(n-1)\frac{8}{9}\left(\frac{3}{4}\right)^{n}.$$

In the limit, we might expect all players of the tournament to have approximately the same number of wins, which tells us that we should have a very large number of kings. Combined with our naive perception of "winners", this has the unfortunate implication that for large tournament sizes, *almost every member is a "winner"*.

To top off this section, we take a look at another way to classify tournaments. This time, instead of trying to shrink tournaments, we look at an even bigger one!

Definition 3.5. We call a tournament $T = (V, E)$ *super-kingable* if there exists a tournament $X = (U, F)$ where T is a subtournament of X and the set of kings of X is exactly V.

Problem 3.13 (2pts)**.** *Prove that if a non-trivial tournament T is super-kingable, then the tournament T itself does not contain an Alexandrian king. (A tournament with a single player is considered trivial.)*

Suppose T is the king subtournament of another tournament W, and assume for the sake of contradiction that T has an Alexandrian king e. Then another member of T, say t, must beat e in two steps. But this implies that L_e is nonempty, so by Problem 3.4, it follows that there is a king of W not in T.

Problem 3.14 (6pts)**.** *Prove that if a non-trivial tournament T contains no Alexandrian king, then T is super-kingable.*

Let $T = (V(T), E(T)) = (\{p_1, \ldots, p_n\}, \{e_1, \ldots, e_k\})$ be such that T does not contain an Alexandrian king, and T' be a copy of T such that $T' = (V(T'), E(T')) = (\{p'_1, \ldots, p'_n\}, \{e'_1, \ldots, e'_k\})$ and $e'_j = (p'_a \to p'_b)$ only if $e_j = (p_a \to p_b)$. Construct a new tournament W with vertex set $V(T) \cup V(T')$ such that $p_i \to p'_j$ only if $i \neq j$, $i, j \in \{1, \ldots, n\}$. We claim that p_i is a king of W for all $i \in \{1, \ldots, n\}$. Note that by construction, p_i beats all p'_j in one step if $j \neq i$, and since $p'_j \to p_j$, it follows that p_i beats all p_j in (at most) two steps. Moreover, since p'_i is not an Alexandrian king by assumption, there exists some $p'_k \in V(W)$ such that $p_i \to p'_k$ and $p'_k \to p'_i$. Next, we claim that p'_i is not a king for all $i \in \{1, \ldots, n\}$. By assumption there exists some p'_j such that $p'_j \to p'_i$ because p'_i is not an Alexandrian king of T'. By construction $p_j \to p_i$. It follows that there is no one- nor two-path from p'_i to p_j, so p'_i is not a king.