1. Using the numbers $3, 3, 8$, and 8 exactly once each, and only the operations of addition, subtraction, multiplication, and division, form an expression equal to 24. You may apply the operations in any order through the use of parentheses. Find and **submit this expression**.

   Answer: $\frac{8}{3-\frac{8}{3}}$ **or** $8 \div (3 - (8 \div 3))$ **or** $8/(3-(8/3))$

   Solution: The expression we get is

   $$24 = \boxed{\frac{8}{3-\frac{8}{3}}}.$$

   Alternatively, we can write this as $8 \div (3 - (8 \div 3))$ or $8/(3-(8/3))$.

2. The Stanford Tree buys 15 eggs to bake a cake. Unfortunately, n of these eggs are stale and one additional egg is rotten. The recipe requires the Tree to choose 3 eggs from the lot. A combination is called *bad* if it contains at least one stale egg and no rotten eggs and *awful* if it contains the rotten egg (even if no other eggs are stale). Let x denote the number of bad combinations and y denote the number of awful combinations. Given that the ratio of x to y is $40 : 13$, compute n.

   Answer: 5

   Solution: The total number of ways to choose 3 eggs from 15 eggs is

   $$\binom{15}{3} = 455.$$

   The number of "bad" combinations is:

   $$x = \binom{14}{3} - \binom{14-n}{3}.$$

   Since an "awful" combination includes the rotten egg, we just choose 2 other eggs from the remaining 14 eggs, including both stale and non-stale eggs. This can be represented by the following:

   $$y = \binom{14}{2} = 91$$

   Therefore, the ratio of x to y is

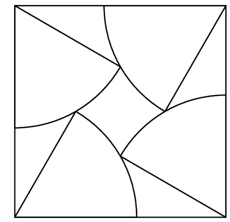
   $$\frac{x}{y} = \frac{\binom{14}{3} - \binom{14-n}{3}}{\binom{14}{2}}$$

   Setting $\frac{x}{y} = \frac{40}{13}$,

   $$\binom{14-n}{3} = 84.$$

   This implies $(14-n)(13-n)(12-n) = 504$, which is satisfied by $n = \boxed{5}$.

3. Four slices of pizza are placed in a square box as shown in the diagram. Each pizza slice is a 60° sector of a circle with radius 1. Compute the side length of the box.



Answer: $\sqrt{3}$

Solution: Take a point where a pizza slice touches another pizza slice, and consider both radii from this point to the centers of the two touching pizza slices. Since both radii have length 1, and the angle that one radii makes with the side of the square has angle $90^\circ - 30^\circ = 60^\circ$. Thus, the two radii, along with the side of the box, form a $30 - 30 - 120$ triangle with short side 1. By Law of Cosines, the pizza box has length $\sqrt{1+1-2(1)(1)\cos(120^\circ)} = \boxed{\sqrt{3}}$. We can also get this answer, if we drop the altitude from the point where these two slices meet. Then we are left with two $30 - 60 - 90$ triangles, and get that the square side length is $2 \cdot \cos(30) = \boxed{\sqrt{3}}$.

4. Compute the unique integer triple (a, b, c) satisfying $0 < a < b < c < 100$ and $a \cdot b + c = 200$ that minimizes $c - a$.

Answer: $(13, 14, 18)$

Solution: The intuition tells us that $C - A$ is minimized when A is maximized. We try to bound using the product AB. Observe that the largest square less than 200 is $14^2 = 196$. So if $A \geq 14$, then there is no solution, because then $AB > 196$ and therefore $C < 4$, which does not satisfy the constraint $A < C$.

Therefore $A \leq 13$. If $A = 13$, then $13B + C = 200$. Then for some $0 < B' < C'$, we write $B = 13 + B'$ and $C = 13 + C'$. Then we have

$$\begin{aligned} 13(13 + B') + 13 + C' &= 200 \\ 169 + 13B' + 13 + C' &= 200 \\ 13B' + C' &= 200 - 169 - 13 \\ &= 18. \end{aligned}$$

Then we see the only solution is $B' = 1$, $C' = 5$. Therefore, $B = 14$, $C = 18$. Now let's confirm this minimizes $A - C$. If $A = 12$, from $A < C$ we have $B \leq 15$. This yields $C \geq 20$, so $C - A \geq 8$. If $A \leq 11$ and $C - A \leq 5$, $AB + C \leq 11 \cdot 15 + 16 = 181 < 200$.

Thus, the final answer is $(A, B, C) = \boxed{(13, 14, 18)}$.

5. Alice has four sticks of lengths $3, 3, 5$, and x. She notices that it is possible to make a cyclic quadrilateral by joining these sticks at their endpoints, and no matter how she makes this cyclic quadrilateral, it always has a diagonal of length 6. Compute x.

Answer: $\frac{27}{5}$

Solution: There are two possible cyclic quadrilateral shapes: one with sides being $3, 3, x, 5$ and the other with sides being $3, x, 3, 5$ (in order).

Since $3 + 3 \leq 6$, in the former the two triangles formed must be $(3, 5, 6)$ and $(3, x, 6)$ by the Triangle Inequality. In the latter, note that in either place the diagonal is placed, the two triangles formed are once more a $(3, 5, 6)$ and a $(3, x, 6)$ triangle. This is because the this configuration is the only other unique configuration and the Triangle Inequality still forces one of the triangles formed to have lengths $(3, 5, 6)$.

Therefore, in the second cyclic quadrilateral, both diagonals are in fact 6. Therefore, by Ptolemy's, we know that $6 \cdot 6 = 3 \cdot 3 + 5 \cdot x$ and rearranging yields $x = \boxed{\frac{27}{5}}$.

6. Archimedes, Basil, Cleopatra, Diophantus, and Euclid each made exactly one claim, listed below in some order:

   1. At least two of the five statements are false.
   2. Each interior angle of a regular pentagon has angle measure $108°$.
   3. 2025 is divisible by 4.
   4. Diophantus said the truth.
   5. Euclid said the truth.

   Pythagoras is attempting to label each claim with the person who made it. A labeling is *determinable* if there exists a **unique** valid assignment of truth values (TRUE or FALSE) to the five claims such that the system of claims is consistent with the truth values. Compute the number of *determinable* labelings.
   For example, consider the labeling of Archimedes to 1, Basil to 2, Cleopatra to 3, Diophantus to 4, and Euclid to 5. Then one valid (but not unique) assignment of truth values is:

$$1.\ \text{TRUE} \quad 2.\ \text{TRUE} \quad 3.\ \text{FALSE} \quad 4.\ \text{TRUE} \quad 5.\ \text{FALSE}.$$

Answer: 66

Solution: Note that statement 1 must always be true, or else we have a paradox: otherwise, both statement 1 and statement 3 are false, making statement 1 true, which is a contradiction. Since statement 2 is always true, and statement 3 is always false, at least one of claims statement 4 and statement 5 has to be false. Let us now do casework based on who said statement 3. If Archimedes, Basil, or Cleopatra said it, statement 4 and statement 5 cannot refer to statement 3, but as one of them has to be false, this is only possible from referring to itself o referring to each other. If they refer to itself, then claims statement 4 and statement 5 has to be said by Euclid and Diophantus, respectively. In this case, Archimedes, Basil, Cleopatra can be labeled in any order to statement 1, statement 2, and statement 3, making it $3! = 6$ configurations in this case. On the other hand, note that statement 4 and statement 5 cannot be both self-referential: if statement 4 is labeled 4, and statement 5 is labeled 5, then $\{\text{FALSE}, \text{TRUE}\}, \{\text{TRUE}, \text{FALSE}\}, \{\text{FALSE}, \text{FALSE}\}$ are all valid assignments to statement 4 and statement 5. As such, in this case, if one of them if self-referential, then the other must refers to either statement 1 or statement 2. There are 3 ways to choose statement 3, 2 ways to choose statement 4 or statement 5 as self referential, 2 ways to choose between statement 1 and statement 2 such that the non-self-referential statement in statement 4 and statement 5 points to, and $2! = 2$ ways to assign the rest. Therefore, the number of configurations in this case is $3 \cdot 2 \cdot 2 \cdot 2 = 24$.

Now for the other case that statement 3 is said by Diophantus or Euclid. Without loss of generality, suppose that Diophantus said statement 3. Then statement 4 points to statement 3 and is therefore always false. statement 5 cannot be self-referential, as in that case it can be either true or false. As such, it must point to statement 1, statement 2, or statement 4 with three choices. The rest of the claims can be said in any order, giving $3(3!) = 18$ ways of arranging. This is likewise true if Euclid said statement 3, making the number of configurations in this case $2 \cdot 18 = 36$. Finally, the answer is the sum of configurations from all the cases: $6 + 24 + 36 = \boxed{66}$.

7. Compute $\sin(\frac{\pi}{24})\sin(\frac{5\pi}{24}) + \sin(\frac{3\pi}{24})\sin(\frac{7\pi}{24}) + \sin(\frac{5\pi}{24})\sin(\frac{9\pi}{24}) + \cdots + \sin(\frac{19\pi}{24})\sin(\frac{23\pi}{24})$.

Answer: $\frac{10\sqrt{3}+\sqrt{6}+\sqrt{2}}{4}$ **or** $\frac{5\sqrt{3}+\sqrt{2+\sqrt{3}}}{2}$

Solution: Let's call the original expression S. In closed form, we have

$$S = \sum_{m=0}^{9} \sin\left(\frac{(2m+1)\pi}{24}\right)\sin\left(\frac{(2m+5)\pi}{24}\right).$$

We use the product-to-sum formula $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ with $A = \frac{(2m+1)\pi}{24}$ and $B = \frac{(2m+5)\pi}{24}$ to get

$$\sin\left(\frac{(2m+1)\pi}{24}\right)\sin\left(\frac{(2m+5)\pi}{24}\right) = \frac{1}{2}\left[\cos\left(\frac{4\pi}{24}\right) - \cos\left(\frac{(4m+6)\pi}{24}\right)\right] = \frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos\left(\frac{(2m+3)\pi}{12}\right)\right].$$

This means that

$$S = \sum_{i=0}^{9} \frac{1}{2}\left[\frac{\sqrt{3}}{2} - \cos\left(\frac{(2m+3)\pi}{12}\right)\right] = \frac{1}{2}\left[5\sqrt{3} - \sum_{i=0}^{9}\cos\left(\frac{(2m+3)\pi}{12}\right)\right].$$

By symmetry,

$$\sum_{i=0}^{9}\cos\left(\frac{(2m+3)\pi}{12}\right) = \sum_{i=0}^{11}\cos\left(\frac{(2m+3)\pi}{12}\right) - \cos\left(\frac{23\pi}{12}\right) - \cos\left(\frac{25\pi}{12}\right) = -2\cos\left(\frac{\pi}{12}\right).$$

Using a half angle identity, we get

$$\cos\left(\frac{\pi}{12}\right) = \sqrt{\frac{1+\cos(\frac{\pi}{6})}{2}} = \sqrt{\frac{\sqrt{3}+2}{4}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Substituting gives a final answer of

$$\frac{1}{2}\left[5\sqrt{3} - \sum_{i=0}^{9}\cos\left(\frac{(2m+3)\pi}{12}\right)\right] = \frac{5\sqrt{3} + \frac{\sqrt{3}+1}{\sqrt{2}}}{2} = \boxed{\frac{10\sqrt{3}+\sqrt{6}+\sqrt{2}}{4}}.$$

8. Compute the number of ordered pairs of positive integers (x, y) with $x + y \leq 20$ for which there exists a positive integer n satisfying $n = \operatorname{lcm}(n, x) - \gcd(n, y)$.

Answer: 36

Solution: Working modulo n, notice that

$$n = \operatorname{lcm}(n, x) - \gcd(n, y) \implies 0 \equiv 0 - \gcd(n, y) \pmod{n}$$
$$\implies \gcd(n, y) \equiv 0 \pmod{n} \implies n \mid \gcd(n, y) \implies$$
$$\gcd(n, y) = n \iff n \mid y$$

Substituting into the original equation yields $\operatorname{lcm}(n, x) = 2n$, so

$$x = \frac{\operatorname{lcm}(n, x) \gcd(n, x)}{n} = 2 \gcd(n, x).$$

If (x, y, n) are such that $n = \operatorname{lcm}(n, x) - \gcd(n, y)$, then define $n' = \gcd(n, x) = \frac{x}{2}$. Note that $n \mid y$ and $n' \mid n$, so $\gcd(n', y) = n'$. Now, we verify that

$$\operatorname{lcm}(n', x) - \gcd(n', y) = \operatorname{lcm}\left(\frac{x}{2}, x\right) - n' = x - n'$$
$$= 2 \gcd(n, x) - \gcd(n, x) = \gcd(n, x) = n'.$$

As such, we can assume $n = \frac{x}{2}$ without missing any pairs (x, y). We then enumerate:

Take $n = 1$. Then, $x = 2$ and $y = 1, 2, \ldots, 18$.

Take $n = 2$. Then, $x = 4$ and $y = 2, 4, \ldots, 16$.

Take $n = 3$. Then, $x = 6$ and $y = 3, 6, 9, 12$

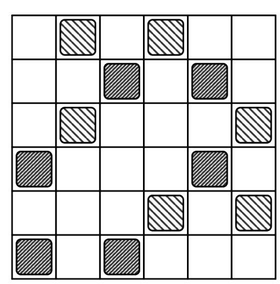
Take $n = 4$. Then, $x = 8$ and $y = 4, 8, 12$.

Take $n = 5$. Then, $x = 10$ and $y = 5, 10$

Take $n = 6$. Then, $x = 12$ and $y = 6$.

For $n \geq 7$, we have $x = 2n \geq 14$ and $n \mid y \implies y \geq 7$, so $x + y \geq 21 > 20$. This gives a final answer of $18 + 8 + 4 + 3 + 2 + 1 = \boxed{36}$.

9. On a 6×6 grid, 12 identical pebbles are placed in the cells such that there are exactly 2 pebbles in every row and every column. Partition the pebbles into groups by declaring that two pebbles belong to the same group if one can reach the other by a sequence of moves between pebbles that lie in the same row or the same column. For example, in the grid shown, there are two groups.



Given a valid arrangement A of these 12 pebbles, define $f(A)$ to be the number of groups in A. Let S be the set of all valid arrangements A. Compute $\sum_{A \in S} 2^{f(A)}$.

Answer: 190800

Solution: Note that every pebble has exactly two neighbors (pebbles in the same row or column), so these groups must be loops. We will color the pebbles in the following manner: Start at one pebble in a group, color it green or red, and move to one of its neighbors, coloring it the opposite color. Continue until all pebbles in the group are colored, then color the rest of groups in the same way.

Note that there are 2 ways to color a group, so there are $2^{f(A)}$ ways to color an arrangement A; hence, we are looking for the number of different possible colorings over all groups.

However, note that in a coloring, the pebbles of one color are all in different rows and columns, so we can interpret it as a permutation of the rows with respect to the columns. For a pair of permutations to be valid, we must have that no two elements in the same positions of these permutations are equal; alternatively, one permutation is a derangement of the other.

Using any way to calculate !6, the number of derangements of a 6-element permutation, the answer is just $(6!)(!6) = 720 \cdot 265 = \boxed{190800}$.

10. For positive integers p and q, there are exactly 14 ordered pairs of real numbers (a, b) with $0 < a, b < 1$ such that both $ap + 3b$ and $2a + bq$ are positive integers. Compute the number of possible ordered pairs (p, q).

Answer: 7

Solution: We consider the conditions into the coordinate plane. $(ap + 3b, 2a + bq)$ cover a parallelogram with vertices $(0, 0), (p, 2), (3, q)$, and $(p + 3, q + 2)$. We are looking for the number of ordered pairs of positive integers (p, q) such that there are exactly 14 of lattice points in the interior of this parallelogram. We may do this using Pick's Theorem: $[\text{Area}] = I + \frac{B}{2} - 1$, where I is the number of interior lattice points and B is the number of boundary lattice points of the parallelogram.
First, we directly compute the area of the parallelogram to be:

$$[\text{Area}] = (p + 3)(q + 2) - 2 \cdot \frac{3(q + 2)}{2} - 2 \cdot \frac{2(p + 3)}{2} = pq - 6$$

We do casework on the number of boundary lattice points. The number of boundary lattice points depends on whether there are any lattice points on the line from $(p, 2)$ to the origin on the line from $(q, 3)$ to the origin. We will need to do casework on the values of $\gcd(p, 2)$ and $\gcd(q, 3)$.
Case 1: $\gcd(p, 2) = 1$ and $\gcd(3, q) = 1$. We need not worry about boundary points in this case, as there are only the four vertices. The equation for this would be

$$pq - 6 = 14 + \frac{4}{2} - 1$$

and using our original conditions, the pairs would be $(p, q) = (3, 7), (21, 1)$. Case 2: $\gcd(p, 2) = 2$ and $\gcd(3, q) = 1$. We need to be more attentive to the side with slope $\frac{2}{p}$. This side passes through one extra point, which is $(\frac{p}{2}, 1)$. Similarly, on the parallel side above, there is an extra point at $(\frac{p+6}{2}, q + 1)$. Our equation would be

$$pq - 6 = 14 + \frac{6}{2} - 1$$

The solutions would be $(p, q) = (2, 11), (22, 1)$. Case 3: $\gcd(p, 2) = 1$ and $\gcd(3, q) = 3$. This is similar to case two, except that we now add four extra boundary points instead of two. The equation is

$$pq - 6 = 14 + \frac{8}{2} - 1$$

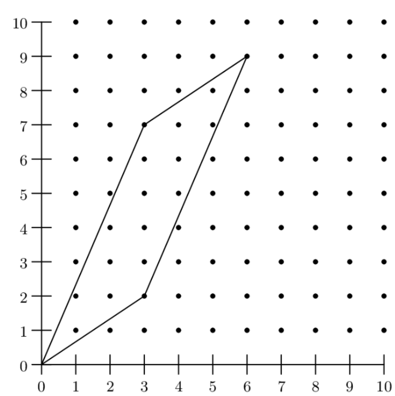
Unfortunately, there are no values of q that satisfy this equation whilst being a multiple of three. Case 4: $\gcd(p, 2) = 2$ and $\gcd(3, q) = 3$. We now have six extra boundary points:

$$pq - 6 = 14 + \frac{10}{2} - 1$$

The solutions here are $(p, q) = (2, 12), (4, 6), (8, 3)$.

In summary, we have seven possible pairs: $(p, q) =$
$(3, 7), (21, 1), (2, 11), (22, 1), (2, 12), (4, 6), (8, 3)$. The answer is $\boxed{7}$.

Plotting $(p, q) = (3, 7)$, for instance, gives us the following (where all points trapped inside represent a pair of $(ap + 3b, 2a + bq)$):



11. Let ω_1 and ω_2 be circles such that ω_2 passes through the center of ω_1 and is also internally tangent to it. In addition, let AB be a chord of ω_1 tangent to ω_2 at some point T. Given that $AT = 9$ and $BT = 16$, compute the radius of ω_1.

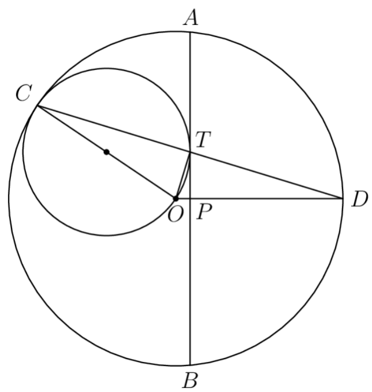
Answer: $\frac{288\sqrt{527}}{527}$

Solution: Let C be the point where ω_2 is tangent to ω_1. Let O be the center of ω_1, and let CT meet ω_1 at point D. Since ω_2 is internally tangent inside circle ω_1, OC is a diameter of ω_2, therefore $\angle OTC = 90^\circ$ since it is an inscribed angle measuring a semicircular arc. As segment OT is perpendicular to chord CD, it must bisect the chord, and therefore $CT = DT$. By Power of a Point on point T, $CT = \sqrt{AT \cdot BT} = \sqrt{9 \cdot 16} = 12$.

We claim that point D must be the midpoint of minor arc AB. We can see this by considering a homothety (dilation) mapping ω_2 to ω_1. This homothety must be centered at their common point at C. Furthermore, this homothety must maps chord AB to some line segment tangent to ω_1 parallel to AB and point T to the tangent point of that new line segment with ω_1. Since this homothety maps T to D, the tangent to ω_1 at D must be parallel to AB. As OD is perpendicular to that tangent, OD must also be perpendicular to AB, and therefore $AD = DB$. This result is often known as the Shooting Lemma.

Let P be the intersection of AB with OD. We have $AP = BP = \frac{25}{2}$ and $TP = \frac{25}{2} - 9 = \frac{7}{2}$. By similarity $\triangle OTD \sim \triangle TDP$, our answer is

$$OD = DT \cdot \frac{DT}{DP} = \frac{12^2}{\sqrt{12^2 - \left(\frac{7}{2}\right)^2}} = \boxed{\frac{288}{\sqrt{527}}} = \boxed{\frac{288\sqrt{527}}{527}}.$$



12. For each positive integer n, define the coloring C_n as a coloring of all lattice points (i.e. points with integer coordinates) on the plane such that any two points $A = (A_x, A_y)$, $B = (B_x, B_y)$ have the same color if and only if $A_x + A_y$ and $B_x + B_y$ have the same remainder when divided by n. A positive integer n is called *broken* if in the coloring C_n there exist two points of different color and Euclidean distance a positive integer divisible by n. Find the sum of all broken positive integers less than or equal to 50.

Answer: 511

Solution:
We claim that n is broken if and only if it the multiple of some 1 modulo 4 prime p.
For any two points, let $X = A_x - B_x$ and $Y = A_y - B_y$. A and B are different colors if and only if $X + Y \not\equiv 0 \bmod n$.
By the provided conditions, whether n is *broken* is equivalent to the existence of points A and B where $X + Y \not\equiv 0 \bmod n$ but $d(A, B) = kn$ where $k \in \mathbb{Z}$.

To prove sufficiency, by Fermat's Theorem on Sum of Two Squares, $p = a^2 + b^2$ for some positive integers a and b. Without loss of generality, assume $a > b$. $n = mp = m(a^2 + b^2)$ for some m. Then, let $A = (0, 0)$ and $B = (m(a^2 - b^2), m(2ab))$. $d(A, B) = \sqrt{m^2((a^2 - b^2)^2 + (2ab)^2)} = m(a^2 + b^2) = n$. However, let $D = (A_x - B_x) + (A_y - B_y) = m(a^2 - b^2 + 2ab) = n + m(2ab - 2b^2) = 2n - m(a^2 + 3b^2 - 2ab) = 2n - m(a - b)^2 - 2mb^2$. As such, $n < D < 2n$ implies that the points A and B have different colors, so n is broken.

To show necessity, assume for the sake of contradiction that there exist *broken* numbers without any $1 \bmod 4$ prime divisor. Let n be the minimal number satisfying this assumption. By definition, there exist points A and B with different colors. Let $X = A_x - B_x, Y = A_y - B_y, n \nmid X^2 + Y^2$. $d(A, B) = \sqrt{X^2 + Y^2} \equiv 0 \bmod n \Longrightarrow X^2 + Y^2 = (kn)^2$ for some $k \in \mathbb{Z}$.
Clearly, 1 is not broken, as all colors are the same. We may assume that there exists some minimal prime p that divides n.
If $p = 2$, n is even, then let $n = 2n' \Longrightarrow X^2 + Y^2 = k^2n^2 = 4k^2(n')^2 \equiv 0 \bmod 4$. Since perfect squares can only be 0 or $1 \bmod 4$, both X and Y have to be even. Let $X = 2X'$, $Y = 2Y'$. Then, $(X')^2 + (Y')^2 = \frac{1}{4}(k^2n^2) = k^2(n')^2$, while $n \nmid X + Y \Longrightarrow n' \nmid X' + Y'$, so n' is also broken with its points being $(0, 0)$ and (X', Y'). This contradicts with the assumption of n's minimality.
Otherwise, $p \equiv 3 \bmod 4$. Let $r = \gcd(X, Y, kn)$. $(\frac{X}{r}, \frac{Y}{r}, \frac{kn}{r})$ is a primitive Pythagorean triplet, so $kn = r(s^2 + t^2)$ for some coprime integers s, t. $r(s^2 + t^2) \equiv 0 \bmod p \Longrightarrow p \mid r$ or $p \mid s^2 + t^2$. If $p \equiv 3 \bmod 4$, $p \mid s^2 + t^2$ implies that both s and t are multiples of p. This fact is by the General Sum of Two Squares' Theorem, derivable through Gaussian integer or quadratic residue. This leads to a contradiction since s and t have to be coprime. Thus, $p \mid r \Longrightarrow p \mid X$ and $p \mid Y$. Similarly, we can check that $n' = \frac{n}{p}$ is a smaller broken number with $A = (0, 0)$ and $B = (\frac{X}{p}, \frac{Y}{p})$.

Finally, we sum $(5 + 10 + \ldots + 50) + (13 + 26 + 39) + (17 + 34) + (29) + (37) + (41) = \boxed{511}$

13. There are thirteen lamps arranged in a circle in the complex plane, with lamp k (for $0 \leq k \leq 12$) located at $z_k = e^{2\pi ik/13}$. Lamp 0 is always on, while lamps 1 to 12 are initially turned off. For each $1 \leq r \leq 12$, toggling lamp r will flip it from on to off (and vice versa) and will also flip its neighbors, excluding lamp 0 if it is a neighbor. Let s be the sum of the positions z_k for lamps which are on. Compute the maximum possible value of $|s|$ over all configurations of switched on lamps reachable by flipping some subset of lamps from the initial state.

Answer: $\frac{\sin(6\pi/13)}{\sin(\pi/13)}$

Solution: We first show that every configuration is reachable by showing that there exist an operation for every lamp 1 to 12 to only flip that lamp without changing others. Note that flipping lamp 1 and lamp 2 flips only lamp 3, and then this operation followed by flipping lamp 4 and 5 flips only lamp 6. Similarly, we can obtain operations to only flip lamp 9 and 12 this way. Doing this from the other end starting with flipping 12 and 11 gives us lamps $10, 7, 4, 1$. Clearly, then, lamps $2, 5, 8, 11$ can be obtained by flipping that lamp, and then flipping the lamps neighboring it.
The maximal scenario is the configuration in which the "turned-on" lamps form a contiguous block. To show this, a proof outline is shown below:

Imagine we have a set of angles corresponding to the lamps that are on. Each angle is of the form $\theta_j = \frac{2\pi j}{13}$, and we want to add the corresponding unit vectors $e^{i\theta_j}$. To make their sum as large as possible, all these vectors have to point in nearly the same direction. If the chosen indices aren't consecutive, there will be a gap between some of the angles, which means that at least one vector points noticeably away from the others and causes more cancellation in the sum. Suppose we have a set $I \subset \{1, 2, \ldots, 12\}$ of size n that is not a block of consecutive numbers. This means that somewhere among our indices there's a "gap"—for instance, we might have j and then $j + 2$ (or larger) instead of j and $j + 1$. Now, consider "shifting" the larger index slightly so that it is closer to its neighbor. When we do this, the corresponding angles become closer together. Because the cosine function is strictly concave on the interval $[0, \pi/2]$ (which is where our small angles lie), the cosine of the average of two angles is strictly greater than the average of their cosines. In our setting, this means that when we "close the gap," the sum of the cosines (which, after all, represent the contributions of the vectors in the direction of the average) increases. By performing such "gap-closing" adjustments one step at a time, we eventually end up with a set I^* where all the indices are consecutive. One can then show that the magnitude of the sum for any non-consecutive set I is strictly less than the magnitude of the sum for the consecutive set I^* of the same size. Now suppose that exactly n consecutive free lamps are turned on. The position of these lamps are given by

$$1, \omega, \omega^2, \ldots, \omega^{n-1}, \quad \text{where } \omega = e^{2\pi i/13}.$$

The sum of these n roots of unity is equal to

$$s_n = 1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \frac{1 - \omega^n}{1 - \omega}.$$

Using the fact that,

$$|\ 1 - e^{i\theta}| = 2\sin\left(\frac{\theta}{2}\right),$$

we can compute:

$$|\ s_n| = \frac{|\ 1 - \omega^n|}{|\ 1 - \omega\ |} = \frac{2\sin(n\pi/13)}{2\sin(\pi/13)} = \frac{\sin(n\pi/13)}{\sin(\pi/13)}.$$

It is easy to see that for $n = 1, 2, \ldots, 12$ the function

$$f(n) = \frac{\sin(n\pi/13)}{\sin(\pi/13)}$$

is increasing for $n \leq 6$ (since $n\pi/13 < \pi/2$) and then decreases by symmetry because

$$\sin\left(\frac{n\pi}{13}\right) = \sin\left(\pi - \frac{n\pi}{13}\right) = \sin\left(\frac{(13-n)\pi}{13}\right).$$

Thus, the maximum occurs at $n = 6$ (or equally $n = 7$, since $\sin(7\pi/13) = \sin(6\pi/13)$). Therefore, the maximal configuration when exactly 6 consecutive lamps are on, and the maximum is:

$$\boxed{\frac{\sin(6\pi/13)}{\sin(\pi/13)}}.$$

14. Let S be the set of all strings of length 15 formed from five 1s, 2s, and 3s. Say a string in S is *threnodic* if:

- No two adjacent characters are the same, and
- Through a sequence of removals of contiguous substrings 123, 231, and 312, the string can be deleted (note that the intermediate strings can have adjacent equal characters).

Compute the number of threnodic strings in S.

Answer: 213

Solution: Modulo 3, we subtract the string 012012012012012 from a threnodic string in S. This transformation changes the string into some string of length 15 of ones, twos and threes with the condition that:

- The substrings 13, 21, and 32 don't appear, and
- with some sequence of removals of the strings $111, 222$, and 333, the string can be deleted.

Characterize inserts by taking some string and inserting characters between two adjacent characters which are identical in a way that preserves the first condition and the third condition. For instance, $111 \mapsto 112223331$ is a valid insert. The number of characters an insert must use is at least 6, and a multiple of 3.

Thus, there can only be 1 or 2 inserts.

We can characterize strings without inserts as strings of the form 11...1, 11...12...2, or 1...12...23...3, where each block of identical digits are a size that is a multiple of 3.

First, without loss, we assume that after the modification, the first character is a 1. Assume there are no inserts. Then, we 2^4 possible strings, since after each block there are only two possible choices for the next block.

Now, consider the case where there is one insert. We can then start with the strings $(111)(111)(111)$, $(111)(111)(222)$, $(111)(222)(222)$, and $(111)(222)(333)$. Since performing an insert between preexisting identical blocks would double count with the no insert case, we have exactly 6 locations where we can perform an insert and exactly one choice for the insert we perform. Thus, in this case we have $6 \cdot 4 = 24$ possible strings if the inserts are 6 big. They can also be 9 or 12 sized, in which case we start with $(111)(111)$ and $(111)(222)$ and (111), respectively. There are only two valid inserts of size 9 and 3 valid inserts of size 12, so then these contribute $4 \cdot 2 \cdot 2 + 2 \cdot 3 \cdot 1$ cases for a total of 46 single inserts.

We have two subcases for the two insert case. We start with 111:

First, the inserts can be nested. Then, we have two choices for where to put the first insert, and then 4 choices for where the second insert goes, for a total of 8 strings.

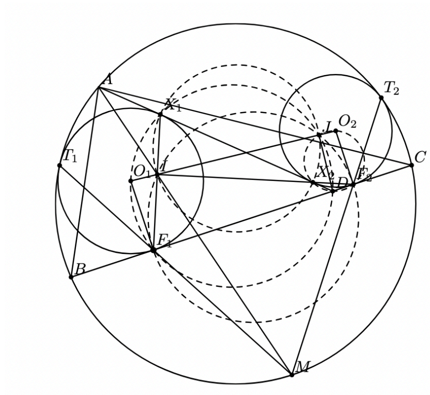
Then, if the inserts aren't nested, there is exactly one way to insert and thus, the two insert case has 9 strings.

In total, for strings starting with any digit there are $3 \cdot (16 + 46 + 9) = \boxed{213}$.

15. Let $\triangle ABC$ be an acute triangle with circumcircle ω_1, and let D be a point on segment BC. Circle ω_2 is tangent to segment AD, segment BD, and ω_1. Circle ω_3 is tangent to segment AD, segment CD, and ω_1, and both circles are on the same side of BC as A. If circles ω_2 and ω_3 have radii 5 and 7, respectively, with centers 13 units apart, compute the sum of all possible lengths of the inradius of $\triangle ABC$.

Answer: $\frac{1980}{169}$

Solution:



Let J be a point on O_1O_2 where $DJ \perp O_1O_2$. Note that $(AX_1T_1), AM, X_1F_1$ are concurrent by Sawayama-Thebault's Theorem; this is because if we intersect AM and X_1F_1, then $\angle T_1AM = \angle T_1X_1F_1$. Note that they also concur at I; this is because $\angle T_1IM = \angle IF_1M$, and using the Incenter-Excenter Lemma in tandem with Power of a Point can prove this. Note that $J = (DF_1X_1) \cap (DF_2X_2)$. Thus J is the center of the spiral similarity sending $X_1F_1 \longmapsto X_2F_2$ by Miquel's Theorem; in particular quadrilateral X_1IX_2J is cyclic, which means that

$$\angle IJX_2 = \angle IX_1X_2 = \angle F_1X_1D = \angle O_2DX_2 = 180^\circ - \angle O_2JX_2$$

Thus points O_1, I, and O_2 collinear. Finally, note that computationally, we know J is the inverse of I with regard to both circles, and so we just need to find the solutions $x \in \mathbb{R}$ to

$$\frac{49}{x} + \frac{25}{13 - x} = 13$$

where $O_1I = x$, which returns two points, the sum of which is $\frac{193}{13}$ by Vieta's. Similarly, since the inradius equals $2(1 - \frac{x}{13}) + 5 = 7 - \frac{2x}{13}$, summing gives $14 - \frac{2}{13} \cdot \frac{193}{13} = \boxed{\frac{1980}{169}}$ which is our answer.